

On the incidence between strata of the Hilbert scheme of points on \mathbb{P}^2

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This talk is based on joint work with Michel Van den Bergh.

In the first part we will recall some basic notions such as Hilbert functions of subschemes of dimension zero, the Hilbert scheme $\text{Hilb}_n(\mathbb{P}^2)$ which parametrizes these subschemes and the stratification corresponding to Hilbert functions. Next we briefly mention a noncommutative version of $\text{Hilb}_n(\mathbb{P}^2)$ and some generalized results. From there we introduce the main question of this talk namely the problem of finding the inclusion relations between closures of strata in the Hilbert scheme of points on \mathbb{P}^2 .

1 Hilbert scheme of points on \mathbb{P}^2

During this talk k is an algebraically closed field of characteristic zero and $A = k[x, y, z]$ is the polynomial ring in three variables viewed as the homogeneous coordinate ring of the projective plane \mathbb{P}^2 . Let $\text{Hilb}_n(\mathbb{P}^2)$ be the Hilbert scheme of zero-dimensional subschemes of degree n in \mathbb{P}^2 . It is well known that this is a smooth connected projective variety of dimension $2n$. Set-theoretically, such a subscheme $X \in \text{Hilb}_n(\mathbb{P}^2)$ consist of n distinct points in the plane. One of the most basic problems is to describe the hypersurfaces that contain X . In particular, we want to know how many hypersurfaces of each degree d contain X . This information is expressed in the *Hilbert function* of X , defined as

$$h_X : \mathbb{N} \rightarrow \mathbb{N} : d \mapsto h_X(d) := \dim (A(X))_d$$

where $A(X)$ denotes the homogeneous coordinate ring of X . In other words, $h_X(d)$ is the rank of the evaluation function in the points of X

$$\theta : A_d \rightarrow k^n$$

These values $h_X(d)$ give information about the position of the points of X . Clearly $h_X(0) = 1$ and $h_X(d) = n$ for sufficiently large values of d relative to n (specifically, for $d \geq n - 1$).

Example 1. The simplest case is where X consists of three points in \mathbb{P}^2 . Then the value $h_X(1)$ tells us whether or not those three points are collinear: we have

$$h_X(1) = \begin{cases} 2 & \text{if the three points are collinear} \\ 3 & \text{if not} \end{cases}$$

and $h_X(d) = 3$ for $d \geq 2$, whatever the position of the points. This follows from the fact that the evaluation function in the three points $A_d \rightarrow k^3$ is surjective, since for any two of the three points there exists a polynomial of degree d vanishing at these two points, but not at the third point.

A numeric function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is said to be a *Hilbert function of degree n* if $\varphi = h_X$ for some subscheme X of dimension zero and degree n . A characterisation of all possible Hilbert functions of degree n was given by Macaulay. Apparently it was Castelnuovo who first recognized the utility of the difference function

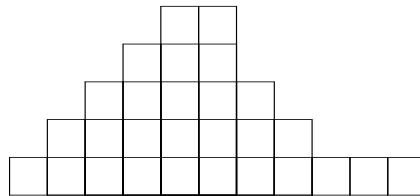
$$s = s_X : \mathbb{N} \rightarrow \mathbb{N} : l \mapsto s_X(d) = h_X(d) - h_X(d-1)$$

which apparently satisfies

$$\begin{cases} s(0) = 1, s(1) = 2, \dots, s(u) = u + 1 \\ s(u) \geq s(u+1) \geq \dots \text{ for some } u \geq 0, \text{ and} \\ s(d) = 0 \text{ for } d \gg 0 \end{cases} \quad (1)$$

Numeric functions $s : \mathbb{N} \rightarrow \mathbb{N}$ for which (1) holds are called *Castelnuovo functions*. It is convenient to visualize them using the graph of a staircase function, as shown in the example below. The number of unit cases in the diagram is called the *weight* of s .

Example 2. $s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^{10}$ is a Castelnuovo polynomial of weight 28. The corresponding diagram is



It is known that a function h is of the form h_X for $X \in \text{Hilb}_n(\mathbb{P}^2)$ if and only if $h(m) = 0$ for $m < 0$ and $h(m) - h(m-1)$ is a Castelnuovo function of weight n . It is natural to put the following ordering on the set of all Hilbert functions of degree n

$$\varphi \leq \psi \text{ if } \varphi(l) \leq \psi(l) \text{ for all } l \in \mathbb{N}$$

The corresponding graph is called the *Hilbert graph* of degree n . As a convention we put the minimal Hilbert series on top.

Example 3. There are three Hilbert functions of degree $n = 5$, namely

$$h_1 : 1 \ 2 \ 3 \ 4 \ 5 \ 5 \ \dots \quad s_3 : 1 \ 1 \ 1 \ 1 \ 1 \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array}$$

corresponds with five collinear points

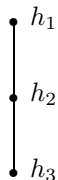
$$h_2 : 1 \ 3 \ 4 \ 5 \ 5 \ \dots \quad s_2 : 1 \ 2 \ 1 \ 1 \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array}$$

five points with exactly four collinear

$$h_3 : 1 \ 3 \ 5 \ 5 \ \dots \quad s_1 : 1 \ 2 \ 2 \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}$$

five points in generic position

The Hilbert graph is



This presentation of Hilbert functions by Castelnuovo stairs has another advantage: given two Hilbert functions φ, ψ of degree n it is easy to decide whether or not $\varphi \leq \psi$, just check if the diagram of φ can be obtained from the diagram of ψ by moving blocks from right to left in such a way that the intermediate graphs are still Castelnuovo stairs.

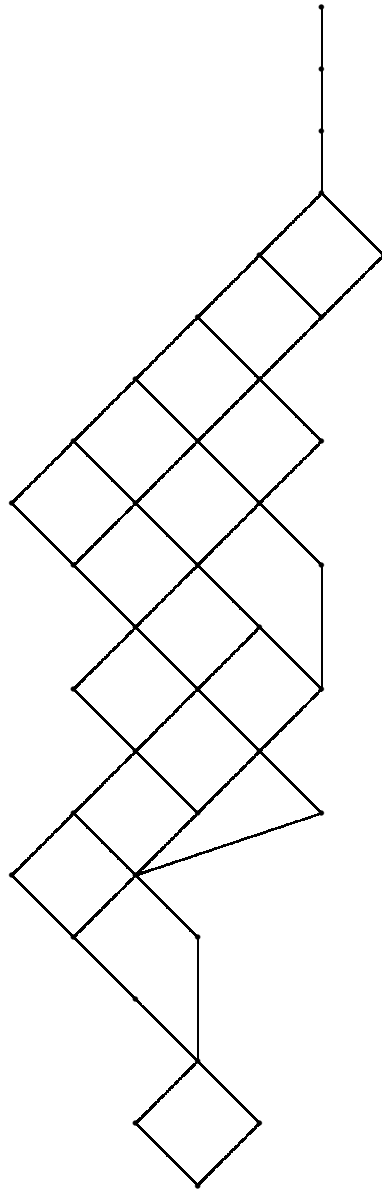
As n becomes larger the number of Hilbert functions increases rapidly and the Hilbert graphs become more complicated. It is easy to see that the number of Hilbert functions of degree n equals the number of partitions of n with odd parts. For instance there are 38 Hilbert functions of degree $n = 17$, the Hilbert graph is plotted below in Example 4. For $n = 100$ the number of Hilbert functions exceeds 450000.

There is a natural stratification of Hilb_n : any Hilbert function φ defines a subscheme H_φ of Hilb_n by

$$H_\varphi = \{X \in \text{Hilb}_n \mid h_X = \varphi\}$$

Gotzmann proved that this strata are smooth, connected and locally closed.

Example 4. The Hilbert graph for $n = 17$ is



2 Noncommutative Hilbert scheme of points

It turns out that some of the above results generalize to certain non-commutative deformations of \mathbb{P}^2 , namely the ones which coordinate ring is a three dimensional Koszul Artin-Schelter regular algebra S . These noncommutative graded rings S are very similar to the commutative polynomial ring $A = k[x, y, z]$. In particular it has the same Hilbert function and the same homological properties. Let $\mathbb{P}_q^2 = \text{Proj } S$ be the corresponding noncommutative \mathbb{P}^2 .

The Hilbert scheme $\text{Hilb}_n(\mathbb{P}_q^2)$ was constructed by Nevins and Stafford (independently by De Naeghel and Van den Bergh for a less general situation). The definition of $\text{Hilb}_n(\mathbb{P}_q^2)$ is not entirely straightforward since in general \mathbb{P}_q^2 will have very few zero-dimensional non-commutative subschemes, so a different approach is needed. It turns out that the correct generalization is to define $\text{Hilb}_n(\mathbb{P}_q^2)$ as the scheme parametrizing the torsion free graded S -modules I of projective dimension one such that

$$h_S(m) - h_I(m) = \dim_k S_m - \dim_k I_m = n \quad \text{for } m \gg 0$$

(in particular I has rank one as S -module). It is easy to see that if S is commutative i.e. $S = k[x, y, z]$ then this condition singles out precisely the graded A -modules which occur as I_X for $X \in \text{Hilb}_n(\mathbb{P}^2)$. We were able to prove

Theorem 1. *There is a bijective correspondence between Castelnuovo polynomials $s(t)$ of weight n and Hilbert series $h_I(t)$ of objects in $\text{Hilb}_n(\mathbb{P}_q^2)$, given by*

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

Moreover, $\text{Hilb}_n(\mathbb{P}_q^2)$ is connected.

Remark 5. The fact that $\text{Hilb}_n(\mathbb{P}_q^2)$ is connected was also proved by Nevins and Stafford for almost all S using deformation theoretic methods and the known commutative case. In case where S is the homogenization of the first Weyl algebra this result was also proved by Wilson. Our proof is intrinsic and entirely different though.

Analogous to the commutative Hilbert scheme we have a stratification on $\text{Hilb}_n(\mathbb{P}_q^2)$ by Hilbert series. We were able to show that as in the commutative case this strata are smooth, connected and locally closed.

As $\text{Hilb}_n(\mathbb{P}^2)$ and $\text{Hilb}_n(\mathbb{P}_q^2)$ have analogous strata, it is natural to ask if the incidence between their strata is also analogous. In other words, does the problem

$$\text{Given two strata } H, H', \text{ when do we have } H \subset \overline{H'}?$$

has the same solution for $\text{Hilb}_n(\mathbb{P}_q^2)$ as it has for $\text{Hilb}_n(\mathbb{P}^2)$? It is evident to consider the commutative case first and learn from its used methods to tackle the noncommutative case. For this talk we will restrict ourselves to \mathbb{P}^2 since the noncommutative case is still in progress.

3 Incidence of strata

As we mentioned above, we will be interested in the following question:

Given two Hilbert functions φ, ψ of degree n , do we have $H_\varphi \subset \overline{H_\psi}$?

In general, this incidence problem is still open. It is linked to the calculation of irreducible components of Brill-Noether strata. Brun, Hirschowitz, Coppo, Walter and Rahavandrainy solved some particular classes of incidence problems. Under a technical condition the incidence problem was solved by Guerimand in the special case where there is no Hilbert function between φ and ψ . Let us recall this result.

If $H_\varphi \subset \overline{H_\psi}$ then it is necessary that

1. $\varphi \leq \psi$. Indeed, for subschemes X, Y of dimension zero and degree n we have (due to semicontinuity)

$$X \subset \overline{\{Y\}} \Rightarrow h_X \leq h_Y$$

2. $\dim H_\varphi < \dim H_\psi$

As shown by numerous examples, the conditions 1,2 are not sufficient. Guerimand introduced a third condition.

For a subscheme X of dimension zero and degree n , define the tangent function $t_X : \mathbb{N} \rightarrow \mathbb{N}$ where

$$t_X(d) = \dim H^0(\mathbb{P}^2, \mathcal{I}_X \otimes \mathcal{T}(d))$$

where \mathcal{T} is the tangent sheaf¹ on \mathbb{P}^2 . By semi-continuity,

$$X \subset \overline{\{Y\}} \Rightarrow t_Y \leq t_X$$

Defining t_φ as t_X where X is the generic point of H_φ , we obtain that if $H_\varphi \subset \overline{H_\psi}$ then

3. $t_\psi \leq t_\varphi$

We have

Theorem 2. (Guerimand) *Let φ, ψ be two Hilbert functions of degree n . Assume that (φ, ψ) has length zero i.e. there is no Hilbert function τ of degree n such that $\varphi < \tau < \psi$. Then, under a technical condition, called 'not of type zero', we have $H_\varphi \subset \overline{H_\psi}$ if and only if*

1. $\varphi \leq \psi$
2. $\dim H_\varphi < \dim H_\psi$
3. $t_\psi \leq t_\varphi$

¹Which is the cokernel of the coordinate map $\mathcal{O} \hookrightarrow \mathcal{O}(1)^3$.

Remark 6. Given two Hilbert functions φ, ψ of degree n , it is easy to see if they have length zero by looking at their Castelnuovo diagrams. Indeed, as we mentioned above we have $\varphi \leq \psi$ if the diagram of ψ may be obtained by moving an upper blocks from right to left in the diagram of φ , in such a way that the intermediate diagrams are valid Castelnuovo diagrams. In particular (φ, ψ) has length zero if there is no way of doing this by moving more than one block.

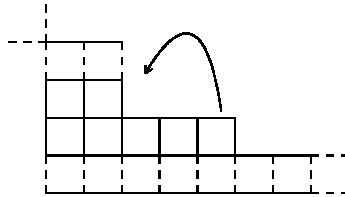
Remark 7. Guerimand proved this theorem using a geometric property called linkage: For positive integers p, q , a pair of subschemes (X, X^*) of dimension zero is said to be on (p, q) if there exist curves C_p, C_q of degree p resp. q such that

$$X \cup X^* = C_p \cap C_q$$

Now if $(X, X^*), (Y, Y^*)$ are both on (p, q) , then under certain conditions on p, q we have the property (called linkage)

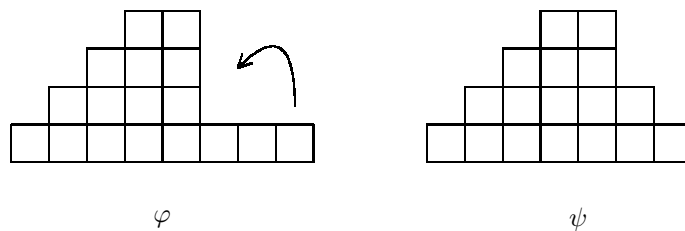
$$X \in \overline{\{Y\}} \Leftrightarrow X^* \in \overline{\{Y^*\}}$$

This method probably does not generalize to the noncommutative case. It was also unknown if Theorem 2 holds in case (φ, ψ) has type zero. A pair (φ, ψ) of Hilbert functions of degree n is of *type zero* if the diagram of φ has the form as shown below, and the diagram of ψ is obtained by moving the upper block as indicated



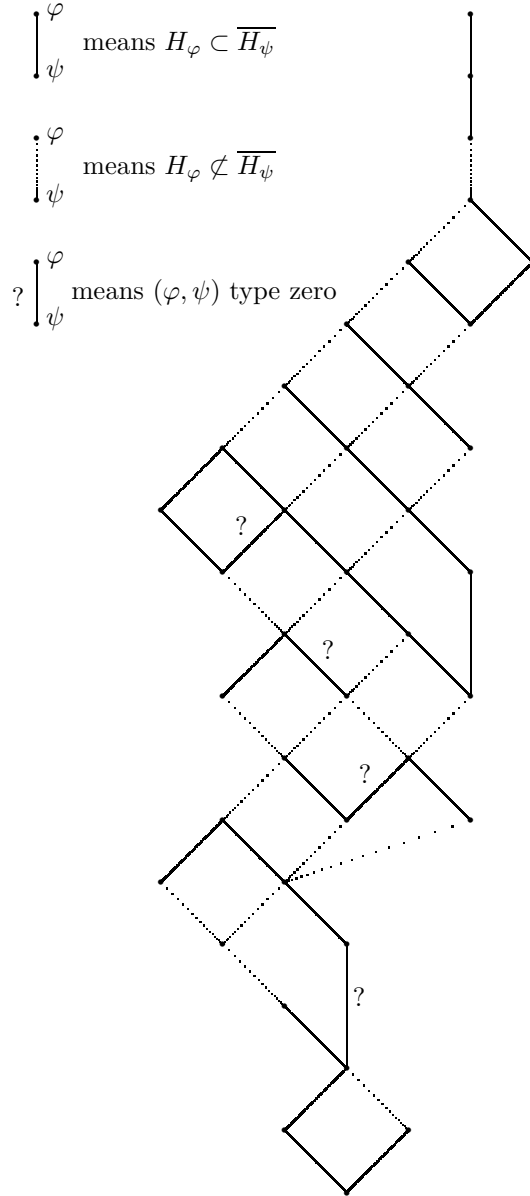
Note that type zero implies length zero and $\varphi \leq \psi$.

Example 8. The Hilbert series (φ, ψ) of degree 17 corresponding to the following diagrams have type zero.

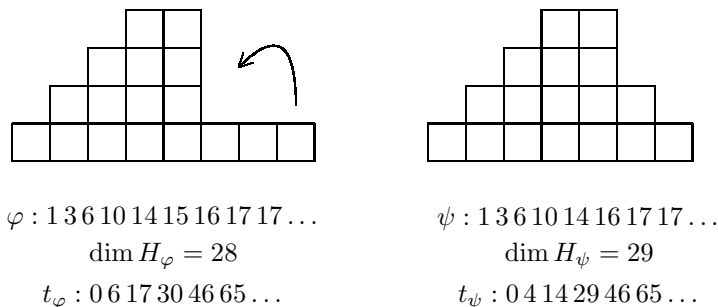


One may ask how many times the pairs (φ, ψ) of type zero occurs. It appears that for $n \gg 0$ the percentage of 'type zero' on the total amount of 'length zero' tends to nearly 7%.

Example 9. Using Theorem 2, the Hilbert graph for $n = 17$ becomes



In case (φ, ψ) has type zero the inclusion relation between the closures of the strata H_φ, H_ψ may be investigated by hand for small n , but was unknown in general. According to Guerimand the first unsolved case is Example 8



Observe that conditions 1,2,3 are satisfied.

Using deformation theory, we were able to reprove Guerimand's result and show that the technical condition 'not of type zero' is not necessary.

Theorem 3. *Let φ, ψ be two Hilbert functions of degree n . Assume that (φ, ψ) has length zero. Then $H_\varphi \subset \overline{H_\psi}$ if and only if*

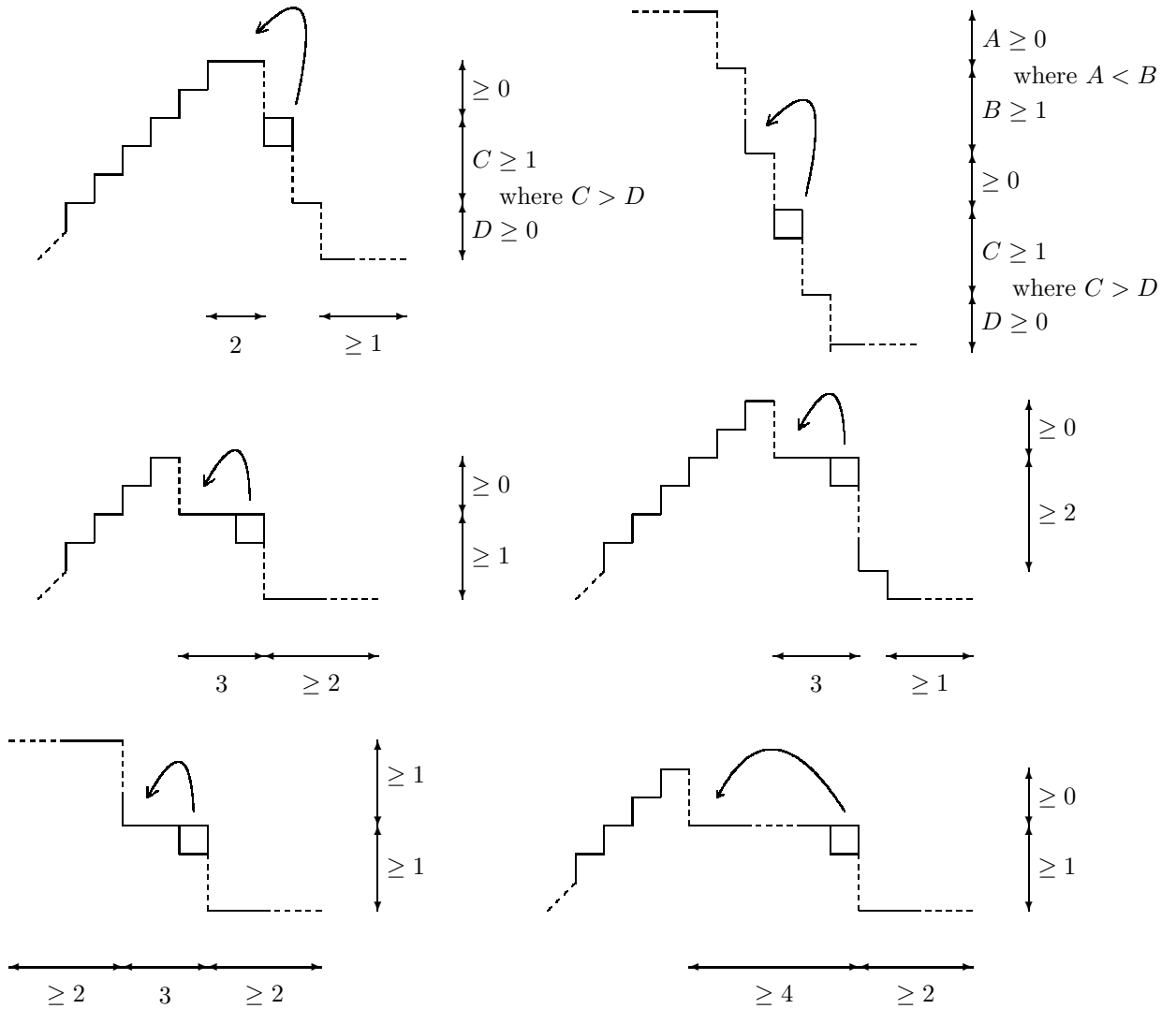
1. $\varphi \leq \psi$
2. $\dim H_\varphi < \dim H_\psi$
3. $t_\psi \leq t_\varphi$

For example, the above unsolved problem (where $n = 17$) now gives $H_\varphi \subset \overline{H_\psi}$. In fact, we proved that all type zero problems are effective.


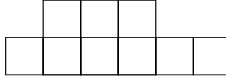
Given two Hilbert functions of degree n which have length zero one is now able to decide the incidence between them, at least in theory, since one has to check the three conditions. For large n quite some computations may be involved to do this. So the question arises for a visual criterion for the conditions in Theorem 3, by which we mean

let (φ, ψ) be a pair of Hilbert series of degree n
 can we decide whether or not $H_\varphi \subset \overline{H_\psi}$
 by looking at the diagrams of φ and ψ ?

Indeed there is such a criterion: $H_\varphi \subset \overline{H_\psi}$ if and only if the Castelnuovo diagram s_φ of φ has one of the following forms, where the diagram s_ψ is obtained by moving the upper block as indicated.



Remark 10. Unfortunately, the conditions 1,2 and 3 are not sufficient in the general case where φ, ψ are arbitrary Hilbert functions of degree n . Guerimand found the following example

	
$\varphi = 1 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 9 \ \dots$	$\psi = 1 \ 3 \ 5 \ 7 \ 8 \ 9 \ 9 \ \dots$
$\dim H_\varphi = 12$	$\dim H_\psi = 13$
$t_\varphi : 1 \ 6 \ 13 \ 22 \ 33 \ 46 \ 62 \ 81 \ \dots$	$t_\psi : 0 \ 3 \ 9 \ 18 \ 30 \ 45 \ 62 \ 81 \ \dots$

Stratum H_φ parametrizes the subschemes of degree 9 containing precisely 8 collinear points.

Stratum H_ψ parametrizes the subschemes of degree 9 containing precisely 6 points on one line D_1 and 3 points on another line D_2 (where D_1 and D_2 are disjoint), these are closed conditions and the generic point of H_φ would have to contain such a configuration, which is not the case.

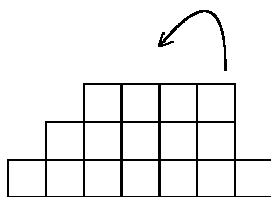
Note that (φ, ψ) does not has length zero.

4 Commutative versus noncommutative

We may use deformation theory to investigate incidence problems for $\text{Hilb}_n(\mathbb{P}_q^2)$. Given Hilbert functions φ, ψ of degree n it is, as in the commutative case, easy to see that the conditions 1,2,3 are necessary such that $H_\varphi \subset \overline{H_\psi}$. Due to the previous we obtain the implication

$$H_\varphi \subset \overline{H_\psi} \text{ in } \text{Hilb}_n(\mathbb{P}_q^2) \Rightarrow H_\varphi \subset \overline{H_\psi} \text{ in } \text{Hilb}_n(\mathbb{P}^2)$$

Although at this moment still in process, we believe that the inverse implication is untrue at least in case the algebra A is generic, i.e. A is a Sklyanin algebra of dimension three where the corresponding translation has infinite order. The first counterexample would be in case of $n = 16$ with Hilbert functions corresponding to the following diagram



The corresponding resolution for a generic ideal I corresponding with the Hilbert function φ is

$$0 \rightarrow A(-7) \oplus A(-8) \rightarrow A(-3) \oplus A(-6)^2 \rightarrow I \rightarrow 0$$