Ideals of cubic algebras and an invariant ring of the Weyl algebra

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1 Introduction and main results

For simplicity we will work over the field of complex numbers $k = \mathbb{C}$. One of the basic examples of a noncommutative noetherian domain is the *first Weyl algebra*

$$A_1 = k\langle x, y\rangle/(xy - yx - 1).$$

It has global dimension one and Gelfand-Kirillov dimension two. It is also well-known that A_1 has no two-sided ideals, however there are plenty one-sided ideals and it is a natural question to describe them. In 1994 Cannings and Holland classified right A_1-ideals by means of the adelic Grassmannian. A few years later Wilson found a relation between the adelic Grassmannian and the so-called *Calogero-Moser spaces* (for $n \in \mathbb{N}$)

$$C_n = \{(\mathbb{X}, \mathbb{Y}) \in M_n(k) \times M_n(k) \mid \operatorname{rank}(\mathbb{Y}\mathbb{X} - \mathbb{X}\mathbb{Y} - \mathbb{I}) \leq 1\}/\operatorname{Gl}_n(k).$$

In 1995 Le Bruyn proposed an alternative classification method based on noncommutative algebraic geometry. His idea was to consider the *homogenized Weyl algebra* H, the algebra obtained by adding a third variable z of degree one to A_1 commuting with x, y and making the relation $xy - yx - 1$ homogeneous

$$H = k\langle x, y, z\rangle/(zx - xz, zy - yz, xy - yx - z^2).$$

The algebra H is a graded k-algebra $H = k \oplus H_1 \oplus H_2 \oplus \dots$ and is seen as a noncommutative analogue of the ordinary polynomial ring in three variables $k[x, y, z]$. In 1979 Beilinson showed the bounded derived category $D^b(\operatorname{coh}(\mathbb{P}^2))$ of coherent sheaves on the projective plane $\mathbb{P}^2 = \operatorname{Proj} k[x, y, z]$ is equivalent to the bounded derived category of representations of a quiver Δ

$$\begin{tikzcd} -2 \arrow[r, shift left=3, "X_{-2}"] \arrow[r, "Y_{-2}"] \arrow[r, shift right=3, "Z_{-2}"] & -1 \arrow[r, shift left=3, "X_{-1}"] \arrow[r, "Y_{-1}"] \arrow[r, shift right=3, "Z_{-1}"] & 0 \end{tikzcd}$$

with relations reflecting the relations in $k[x,y,z]$

$$\begin{cases} Y_{-1}X_{-2} = X_{-1}Y_{-2} \\ Z_{-1}Y_{-2} = Y_{-1}Z_{-2} \\ X_{-1}Z_{-2} = Z_{-1}X_{-2} \end{cases}$$

One may then use this derived equivalence to describe (stable) vector bundles over $\mathbb{P}^2$ in terms of linear algebra.

Le Bruyn observed that this framework does survive when replacing $k[x,y,z]$ with the homogenized Weyl algebra H. Consider $H[z^{-1}]_0$, the degree zero part of the localization $H[z^{-1}]$ at the powers of the central element z. We have $A_1 = H[z^{-1}]_0$. Thus right ideals of A_1 (endowed with a filtration) correspond to graded right ideals I of H which are *reflexive* i.e. $I^{**} = I$ where $I^* = \underline{\operatorname{Hom}}_A(I,H)$ is the graded dual of I. Reflexive graded right ideals of H then correspond to "line bundles" on a "noncommutative plane" $\mathbb{P}^2_{\mathrm{q}} = \operatorname{Proj} H$ in the sense of Artin and Zhang (see below for the exact definitions) and by a likewise derived equivalence they correspond to certain stable representations of the same quiver Δ but with relations reflecting the relations in H.

In 2002 these ideas were worked out by Berest and Wilson to obtain

Theorem A. *Let $R(A_1)$ be the set of isomorphism classes of right A_1-ideals. Then $G = \operatorname{Aut}(A_1)$ has a natural action on $R(A_1)$, where*

- *the orbits of the G-action on $R(A_1)$ are indexed*[1] *by $\mathbb{N}$,*
- *the orbit corresponding to $n \in \mathbb{N}$ is in natural bijection with the n-th Calogero-Moser space*

  $$C_n = \{(\mathbb{X},\mathbb{Y}) \in M_n(k) \times M_n(k) \mid \operatorname{rk}(\mathbb{YX} - \mathbb{XY} - \mathbb{I}) \leq 1\} / \operatorname{Gl}_n(k)$$

  where $\operatorname{Gl}_n(k)$ acts by simultaneous conjugation $(g\mathbb{X}g^{-1}, g\mathbb{Y}g^{-1})$. The C_n are smooth connected affine varieties dimension $2n$.

There are many more k-algebras having the same homological properties like the homogenized Weyl algebra. In particular H is a so-called *three dimensional Artin-Schelter algebra*, which is by definition a graded k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \dots$ satisfying

(i) A has global dimension 3;

(ii) A has polynomial growth i.e. there are positive real numbers c, e such that $\dim_k A_n \leq cn^e$ for all positive integers n;

(iii) A is Gorenstein, meaning there is an integer l such that

$$\underline{\operatorname{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_A k(l) & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

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[1] The fact that $R(A_1)/G \cong \mathbb{N}$ has also been proved by Kouakou in his PhD-thesis.

This class of graded algebras was introduced by Artin and Schelter in 1986 and classified a few years later by Artin, Tate and Van den Bergh (generated in degree one) and Stephenson (general case). They are all noetherian domains of Gelfand-Kirillov dimension three and have all expected nice homological properties. Let us further assume A is generated in degree one. It turns out there are two possibilities for such an algebra A (done by Artin and Schelter)

- k_A has a minimal resolution of the form

  $$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

  Thus A has three generators and three defining homogeneous relations in degree two. We say A is *quadratic*. The Hilbert series of A is given by $h_A(t) := \sum_n \dim_k A_n t^n = 1 + 3t + 6t^2 + 10t^3 + \cdots = (1-t)^{-3}$.

- k_A has a minimal resolution of the form

  $$0 \to A(-4) \to A(-3)^2 \to A(-1)^2 \to A \to k_A \to 0$$

  Now A has two generators and two defining homogeneous relations in degree three. We say A is *cubic*. The Hilbert series of A is given by $h_A(t) = \sum_n \dim_k A_n t^n = 1 + 2t + 4t^2 + 6t^3 + \cdots = (1-t)^{-2}(1-t^2)^{-1}$.

The generic class of quadratic and cubic Artin-Schelter algebras are usually called *type A-algebras*, in which case the relations are respectively given by

$$\begin{cases} ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \\ axy + byx + cz^2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} ay^2x + byxy + axy^2 + cx^3 = 0 \\ ax^2y + bxyx + ayx^2 + cy^3 = 0 \end{cases} \tag{1.1}$$

where $a, b, c \in k$ are generic scalars.

It was shown by Artin, Tate and Van den Bergh that a quadratic or cubic Artin-Schelter algebra A is completely determined by a triple (E, σ, j), depending on A, where either

- $j : E \cong \mathbb{P}^2$ if A is quadratic, resp. $j : E \cong \mathbb{P}^1 \times \mathbb{P}^1$ if A is cubic; or
- $j : E \hookrightarrow \mathbb{P}^2$ is an embedding of a divisor E of degree three if A is quadratic, resp. $j : E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ where E is a divisor of bidegree $(2, 2)$ if A is cubic

and $\sigma \in \mathrm{Aut}(E)$. In the first case we say A is *linear*, otherwise A is called *elliptic*. If A is of type A and the divisor E is a smooth elliptic curve (this is the generic case) then we say A is of *generic type A*. In that case σ is a translation on E. Quadratic Artin-Schelter algebras of generic type A are also called *three dimensional Sklyanin algebras*.

For a linear quadratic resp. cubic Artin-Schelter algebra its reflexive graded right ideals just correspond to the (classical) line bundles on $\mathbb{P}^2$ resp. $\mathbb{P}^1 \times \mathbb{P}^1$.

In 2004, Van den Bergh and the first author generalized the ideas of Le Bruyn and Berest and Wilson to obtain

Theorem B. *Let A be an elliptic quadratic Artin-Schelter algebra for which σ has infinite order. Let $R(A)$ be the set of reflexive graded right A-ideals, considered up to isomorphism and shift of grading. There exist smooth locally closed varieties D_n of dimension $2n$ such that $R(A)$ is naturally in bijection with $\coprod_{n\in\mathbb{N}} D_n$.*

If in addition A is of generic type A i.e. A is a three dimensional Sklyanin algebra then the varieties D_n are affine.

In particular D_0 is a point and D_1 is the complement of E under a natural embedding in $\mathbb{P}^2$. For any $n \in \mathbb{N}$ there is an explicit description of D_n.

A result similar to Theorem 1 was proved by Nevins and Stafford without the restriction on the order of σ, but without the explicit description and the affineness of the varieties D_n. They proved D_n is connected as well.

The aim of this talk is to point out that by using the same methods one obtains a result similar to Theorem B for cubic Artin-Schelter algebras.

Let A be a cubic Artin-Schelter algebra and let $R(A)$ denote the set of reflexive graded right A-ideals, considered up to isomorphism and shift of grading. Define $N = \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$.

Theorem 1. *Let A be an elliptic cubic Artin-Schelter regular algebra for which σ has infinite order. Then for $(n_e, n_o) \in N$ there exists a smooth locally closed variety $D_{(n_e,n_o)}$ of dimension $2(n_e - (n_e - n_o)^2)$ such that $R(A)$ is in natural bijection with $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$.*

If in addition A is of generic type A then the varieties $D_{(n_e,n_o)}$ are affine.

In particular $D_{(0,0)}$ is a point and $D_{(1,1)}$ is the complement of E under a natural embedding in $\mathbb{P}^1 \times \mathbb{P}^1$. In fact $D_{(n_e,n_o)}$ is a point whenever $n_e = (n_e - n_o)^2$.

A crucial part of the proof of Theorem 1 consists in showing that the spaces $D_{(n_e,n_o)}$ are actually nonempty for $(n_e, n_o) \in N$. In contrast to quadratic Artin-Schelter regular algebras this is not entirely straightforward. We have shown this by characterizing the appearing Hilbert series for objects in $R(A)$. We will come back on this at the end of this talk.

As an application, consider the *enveloping algebra of the Heisenberg-Lie algebra*

$$H_c = k\langle x, y, z\rangle/(yz - zy, xz - zx, xy - yx - z) = k\langle x, y\rangle/([y, [y, x]], [x, [x, y]])$$

where $[a, b] = ab - ba$. The graded algebra H_c is a cubic Artin-Schelter algebra.

Consider $H_c[z^{-1}]_0$, the degree zero part of the localization $H_c[z^{-1}]$ at the powers of the central element $z = xy - yx$. We have $A_1^{\langle\varphi\rangle} = H[z^{-1}]_0$, the algebra of invariants of the first Weyl algebra $A_1 = k\langle x, y\rangle/(xy - yx - 1)$ under the automorphism φ defined by $\varphi(x) = -x$, $\varphi(y) = -y$. Right ideals of $A_1^{\langle\varphi\rangle}$ (endowed with a filtration) correspond to reflexive graded right ideals ideals of H_c. Theorem 1 and further investigation of the varieties $D_{(n_e,n_o)}$ leads to

Theorem 2. *The set* $R(A_1^{\langle\varphi\rangle})$ *of isomorphism classes of right* $A_1^{\langle\varphi\rangle}$*-ideals is in natural bijection with the points of* $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$ *where*

$$D_{(n_e,n_o)} = \{(\mathbb{X},\mathbb{Y},\mathbb{X}',\mathbb{Y}') \in M_{n_e\times n_o}(k)^2 \times M_{n_o\times n_e}(k)^2 \mid \mathbb{Y}'\mathbb{X} - \mathbb{X}'\mathbb{Y} = \mathbb{I} \text{ and } \operatorname{rank}(\mathbb{Y}\mathbb{X}' - \mathbb{X}\mathbb{Y}' - \mathbb{I}) \leq 1\}/ \operatorname{Gl}_{n_e}(k) \times \operatorname{Gl}_{n_o}(k)$$

where $\operatorname{Gl}_{n_e}(k) \times \operatorname{Gl}_{n_o}(k)$ *acts by conjugation* $(g\mathbb{X}h^{-1}, g\mathbb{Y}h^{-1}, h\mathbb{X}'g^{-1}, h\mathbb{Y}'g^{-1})$*. The varieties* $D_{(n_e,n_o)}$ *are smooth affine varieties dimension* $2(n_e - (n_e - n_o)^2)$*.*

Comparing with Theorem A it would be interesting to see if the orbits of $R(A_1^{\langle\varphi\rangle})$ under the automorphism group $\operatorname{Aut}(A_1^{\langle\varphi\rangle})$ are in bijection to $D_{(n_e,n_o)}$.

2 Noncommutative quadrics

Let A denote a cubic Artin-Schelter algebra. Thus $A = k \oplus A_1 \oplus A_2 \oplus \dots$ is a graded k-algebra with Hilbert series

$$h_A(t) = \sum_n \dim_k A_n t^n = \frac{1}{(1-t)^2(1-t^2)} = 1 + 2t + 4t^2 + 6t^3 + 9t^4 + 12t^5 + \dots$$

Let M denote a graded right A-module i.e. $M = \oplus_{i\in\mathbb{Z}} M_i$ is a right A-module and $M_iA_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. For any integer n we write $M(n)$ for the graded right A-module that is equal to M with its original A action, but which is graded by $M(n)_i := M_{n+i}$. We write

- $\operatorname{GrMod}(A)$ for the category of graded right A-modules with morphisms the A-module maps that preserve degree,
- $\operatorname{Tors}(A) \subset \operatorname{GrMod}(A)$ for the full subcategory consisting of the direct limits of finite dimensional graded right A-modules,
- $\operatorname{Tails}(A)$ for the quotient category $\operatorname{GrMod}(A)/\operatorname{Tors}(A)$ and $\pi : \operatorname{GrMod}(A) \to \operatorname{Tails}(A)$ for the (exact) quotient functor
- $\operatorname{grmod}(A)$, $\operatorname{tors}(A)$, $\operatorname{tails}(A)$ for the corresponding full subcategories consisting of the noetherian objects.

Objects in $\operatorname{Tails}(A)$ will be denoted by script letters, like $\mathcal{M}$. We put $\mathcal{O} = \pi A$. The shift functor $M \mapsto M(1)$ on $\operatorname{GrMod}(A)$ induces an automorphism $\operatorname{sh} : \mathcal{M} \mapsto \mathcal{M}(1)$ on $\operatorname{Tails}(A)$.

Following Artin and Zhang, we define the non-commutative projective scheme as the triple

$$\operatorname{Proj} A := (\operatorname{tails}(A), \mathcal{O}, \operatorname{sh})$$

The Hilbert series of the Veronese subalgebra $A^{(2)} = k \oplus A_2 \oplus A_4 \oplus \dots$ of A is the same as that of the homogeneous coordinate ring $k[x_0, x_1, x_2, x_3]/(x_0x_1 - x_2x_3)$ of a quadratic surface (quadric) in $\mathbb{P}^3$. Since $\operatorname{Tails}(A) \cong \operatorname{Tails}(A^{(2)})$ has cohomological dimension two we therefore think of $\operatorname{Proj} A$ as a *quantum quadric*, a noncommutative analogue of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. We write $X = \operatorname{Proj} A$ and put

$$\operatorname{Qcoh}(X) := \operatorname{Tails}(A)$$
$$\operatorname{coh}(X) := \operatorname{tails}(A)$$

thinking of them as the (quasi)coherent sheaves on X, even though they are not really sheaves.

Recall from the Introduction that A is completely determined by a triple (E, σ, j) where E is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a divisor of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us briefly point out how this is done (due to Artin, Tate and Van den Bergh).

One consideres a special kind of modules over A, called *point modules*. These are by definition cyclic graded right A-modules P generated in degree zero with Hilbert series

$$h_P(t) = \sum_n \dim_k P_n t^n = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$$

Let $P = \oplus_i P_i$ be such a point module. Choosing a basis $e_0, e_1, e_2, \dots$ in each k-vector space $P_0, P_1, P_2, \dots$ we find

$$\begin{cases} e_0 x = \alpha_0 e_1 \\ e_0 y = \beta_0 e_1 \end{cases}, \quad \begin{cases} e_1 x = \alpha_1 e_2 \\ e_1 y = \beta_1 e_2 \end{cases}, \quad \begin{cases} e_2 x = \alpha_2 e_3 \\ e_2 y = \beta_2 e_3 \end{cases}, \dots$$

for some scalars $\alpha_i, \beta_i \in k$. Each relation between the generators of A must kill e_0 (or any e_i) and this leads to an equation in $(\alpha_0, \beta_0), (\alpha_1, \beta_1)$. Let E denote the zero locus in $\mathbb{P}^1 \times \mathbb{P}^1$ of that equation (this is the same E as in the Introduction).

Example 2.1. Let us first consider the generic case i.e. A is of type A. Using the relations (1.1) we find

$$\begin{pmatrix} a\beta_0\beta_1 + c\alpha_0\alpha_1 & b\beta_0\alpha_1 + a\alpha_0\beta_1 \\ a\beta_0\alpha_1 + b\alpha_0\beta_1 & a\alpha_0\alpha_1 + c\beta_0\beta_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = 0$$

hence (by taking the determinant) we deduce that E is given by

$$(c^2 - b^2)\alpha_0\beta_0\alpha_1\beta_1 + a\alpha_0^2(c\alpha_1^2 - b\beta_1^2) + a\beta_0^2(c\beta_1^2 - b\alpha_1^2) = 0$$

Generically E will be a smooth elliptic curve.

Second, if $A = H_c$ is the enveloping algebra of the Heisenberg Lie algebra we find

$$\begin{pmatrix} \beta_0\beta_1 & \alpha_0\beta_1 - 2b\beta_0\alpha_1 \\ \beta_0\alpha_1 - 2\alpha_0\beta_1 & \alpha_0\alpha_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = 0$$

and E is given by $(\alpha_0\beta_1 - \beta_0\alpha_1)^2 = 0$, the double diagonal $2D$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Thus to any pointmodule P over A we may associate a closed point $p \in E$ and this assignment $P \mapsto p$ turns out to be bijective.

Since $P' := P(1)_{\geq 0} = P_1 \oplus P_2 \oplus \dots$ is also a pointmodule over A, it determines also a closed point $p' \in E$. Defining $\sigma : E \to E : p \mapsto p'$ one may show that σ is an automorphism of E.

Associated to the geometric data (E, σ, j) is the so-called "twisted" homogeneous coordinate ring B. If A is linear then $B \cong A$, and if A is elliptic there is a normal element g of degree 4 of A such that $B \cong A/gA$. Though the structure of $\operatorname{Proj} A$ is somewhat obscure, that of $\operatorname{Proj} B$ is well understood: there is an equivalence of categories

$$\begin{tikzcd} \mathrm{Tails}(B) \arrow[r, shift left, "(\tilde{-})"] & \mathrm{Qcoh}(E) \arrow[l, shift left, "\Gamma_*"] \end{tikzcd}$$

Combining with the relation between B and A this gives us a pair of adjoint functors i^*, i_*

$$\begin{tikzcd} \mathrm{Qcoh}(X) \arrow[r, shift left, "-\otimes_A B"] \arrow[rr, bend left, "i^*"] & \mathrm{Tails}(B) \arrow[l, shift left, "(-)_A"] \arrow[r, shift left, "(\tilde{-})"] & \mathrm{Qcoh}(E) \arrow[l, shift left, "\Gamma_*"] \arrow[ll, bend left, "i_*"] \end{tikzcd}$$

We refer to right exact functor i^* as the *restriction functor*. Note i_* is exact.

3 From reflexive ideals to normalized line bundles

Let A denote a cubic Artin-Schelter algebra and $X = \operatorname{Proj} A$. Let I be a graded right A-ideal i.e. $I \in \mathrm{grmod}(A)$ and $I \subset A$ by which we mean $I_m \subset A_m$ for all integers m. It is intuitively clear that the difference function $m \mapsto \dim_k A_m - \dim_k I_m$ is linear for $m \gg 0$. Indeed, elementary calculations show that for any graded right A-module J of rank one there is a (unique) shift $d \in \mathbb{Z}$ for which

$$\dim_k A_m - \dim_k J(d)_m = \begin{cases} n_e & \text{if } m \text{ is even,} \\ n_o & \text{if } m \text{ is odd} \end{cases} \quad \text{for } m \gg 0.$$

for some integers n_e, n_o. We say $J(d)$ is normalized and refer to (n_e, n_o) as the invariants of J (and $J(d)$). We will see below that these numbers n_e, n_o are actually positive.

For any integers $n_e, n_o \in \mathbb{Z}$ we denote by $R_{(n_e,n_o)}(A)$ the full subcategory of $\mathrm{grmod}(A)$ with objects

$$R_{(n_e,n_o)}(A) = \{\text{normalized reflexive graded right } A\text{-modules with invariants } (n_e, n_o)\}$$

We obtain a natural bijection between the set

$$R(A) = \{\text{reflexive graded right } A\text{-ideals}\}/\text{iso, shift}$$

and the isomorphism classes in the category $\coprod_{(n_e,n_o)\in\mathbb{Z}^2} R_{(n_e,n_o)}(A)$.

Next, we consider the exact quotient functor $\pi : \operatorname{grmod}(A) \to \operatorname{coh}(X) : M \mapsto \mathcal{M}$. The image of a (normalized) rank one module is called a (normalized) line bundle, and the category $R_{(n_e,n_o)}(A)$ is equivalent with the full subcategory $\mathcal{R}_{(n_e,n_o)}(X)$ of $\operatorname{coh}(X)$ with objects

$$\mathcal{R}_{(n_e,n_o)}(X) = \{\text{normalized line bundles on } X \text{ with invariants } (n_e,n_o)\}$$

It is not hard to verify that $R_{(n_e,n_o)}(A)$ and $\mathcal{R}_{(n_e,n_o)}(X)$ are actually groupoids.

The next step is to compute (partially) the cohomology groups of an object $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$

$$H^i(X,\mathcal{I}) := \operatorname{Ext}^i_X(\mathcal{O},\mathcal{I}).$$

The result is (for $\mathcal{I} \ncong \mathcal{O}$)

l	...	-6	-5	-4	-3	-2	-1	0	1	2	...
$\dim_k H^0(X,\mathcal{I}(l))$	...	0	0	0	0	0	0	0	$*$	$*$	...
$\dim_k H^1(X,\mathcal{I}(l))$	...	$*$	$*$	n_e-1	n_o	n_e	n_o	n_e-1	$*$	$*$	...
$\dim_k H^2(X,\mathcal{I}(l))$	...	$*$	$*$	0	0	0	0	0	0	0	...

From which we immediately deduce $n_e \geq 1$, $n_o \geq 0$. In particular one may show $\mathcal{R}_{(0,0)}(X) = \{\mathcal{O}\}$. At this point one may be tempted to think there are two independent parameters $n_e, n_o \in \mathbb{N}$ associated to an object $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$. However, an elementary computation on the Euler forms show

$$\dim_k \operatorname{Ext}^1_X(\mathcal{I},\mathcal{I}) = 2(n_e - (n_e - n_o)^2)$$

hence for any integers n_e, n_o

$$\mathcal{R}_{(n_e,n_o)}(X) \neq \emptyset \Rightarrow (n_e,n_o) \in N \tag{3.1}$$

where $N = \{(n_e,n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$ is as in the Introduction. A detailed study of the possible Hilbert functions of reflexive graded right A-ideals reveals the converse implication of (3.1) is also true. See Section 7.

4 From normalized line bundles to quiver representations

Let A denote a cubic Artin-Schelter algebra and $X = \operatorname{Proj} A$. There is an equivalence of bounded derived categories (follows from a more general theorem of Bondal)

$$\begin{tikzcd} \mathrm{D}^b(\operatorname{coh}(X)) \arrow[r, shift left, "\operatorname{RHom}_X(\mathcal{E},-)"] & \mathrm{D}^b(\operatorname{mod}(\Gamma)) \arrow[l, shift left, "-\overset{\mathbf{L}}{\otimes}_\Gamma \mathcal{E}"] \end{tikzcd} \tag{4.1}$$

where $\mathcal{E} = \mathcal{O}(3) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and Γ is the quiver

$$\begin{tikzcd} -3 \arrow[r, shift left, "X_{-3}"] \arrow[r, shift right, "Y_{-3}"'] & -2 \arrow[r, shift left, "X_{-2}"] \arrow[r, shift right, "Y_{-2}"'] & -1 \arrow[r, shift left, "X_{-1}"] \arrow[r, shift right, "Y_{-1}"'] & 0 \end{tikzcd} \tag{4.2}$$

with relations R reflecting the relations of A.

We would like to understand the image of $\mathcal{R}_{(n_e,n_o)}$ under the equivalence (4.1). So let $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$ and consider $\mathcal{I}$ as a complex in $\mathrm{D}^b(\mathrm{coh}(X))$ of degree zero. Due to the previous, the image of this complex is concentrated in degree one

$$\mathrm{RHom}_X(\mathcal{E}, \mathcal{I}) = M[-1]$$

where $M = \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{I})$. By functoriality, multiplication by $x \in A_1$ induces linear maps, represented by (left) matrix multiplications [2]

$$H^1(X, \mathcal{I}(-3)) \xrightarrow{X\cdot} H^1(X, \mathcal{I}(-2)) \xrightarrow{X'\cdot} H^1(X, \mathcal{I}(-1)) \xrightarrow{X''\cdot} H^1(X, \mathcal{I})$$

and similar for $y \in A_1$. Thus M given by the following representation of Γ

$$\begin{tikzcd} H^1(X, \mathcal{I}(-3)) \arrow[r, shift left, "X\cdot"] \arrow[r, shift right, "Y\cdot"'] & H^1(X, \mathcal{I}(-2)) \arrow[r, shift left, "X'\cdot"] \arrow[r, shift right, "Y'\cdot"'] & H^1(X, \mathcal{I}(-1)) \arrow[r, shift left, "X''\cdot"] \arrow[r, shift right, "Y''\cdot"'] & H^1(X, \mathcal{I}) \end{tikzcd}$$

with dimension vector $\underline{\dim} M = (n_o, n_e, n_o, n_e - 1)$, and satisfying the relations of the quiver Γ. For example, if A is of type A then

$$\begin{pmatrix} X'' & Y'' \end{pmatrix} \cdot \begin{pmatrix} aY'Y + cX'X & bX'Y + aY'X \\ bY'X + aX'Y & aX'X + cY'Y \end{pmatrix} = 0 \tag{4.3}$$

We now want to see how the reflexivity of $\mathcal{I}$ is translated through the derived equivalence. Consider a point module P over A, $\mathcal{P} = \pi P$. As the cohomology groups $H^i(X, \mathcal{P}) := \mathrm{Ext}^i_X(\mathcal{O}, \mathcal{P})$ of $\mathcal{P}$ are given by

l	$\ldots$	-6	-5	-4	-3	-2	-1	0	1	2	$\ldots$
$\dim_k H^0(X, \mathcal{I}(l))$	$\ldots$	1	1	1	1	1	1	1	1	1	$\ldots$
$\dim_k H^1(X, \mathcal{I}(l))$	$\ldots$	0	0	0	0	0	0	0	0	0	$\ldots$
$\dim_k H^2(X, \mathcal{I}(l))$	$\ldots$	0	0	0	0	0	0	0	0	0	$\ldots$

one sees that $\mathcal{P}$ determines the following representation p of Γ

$$\begin{tikzcd} k \arrow[r, shift left, "\alpha_{-3}\cdot"] \arrow[r, shift right, "\beta_{-3}\cdot"'] & k \arrow[r, shift left, "\alpha_{-2}\cdot"] \arrow[r, shift right, "\beta_{-2}\cdot"'] & k \arrow[r, shift left, "\alpha_{-1}\cdot"] \arrow[r, shift right, "\beta_{-1}\cdot"'] & k \end{tikzcd}$$

where $p = ((\alpha_0, \beta_0), (\alpha_1, \beta_1)) \in E$ is the closed point[3] corresponding to the point module P and $(\alpha_i, \beta_i) = \mathrm{pr}_1\, \sigma^i p$.

---

[2] There should be no confusion between the linear map $X\cdot$ and the quantum quadric $X = \mathrm{Proj}\, A$.

[3] There should be no confusion between the closed point $p \in E$ and its corresponding representation $p \in \mathrm{mod}(\Gamma)$.

Saying $\mathcal{I}$ reflexive means $\operatorname{Ext}^1_X(\mathcal{P},\mathcal{I})=0$. By the derived equivalence (4.1)

$$\begin{aligned}\operatorname{Ext}^1_X(\mathcal{P},\mathcal{I}) = &\ H^0(\operatorname{RHom}_X(\mathcal{P},\mathcal{I}[1]))\\ \cong &\ H^0(\operatorname{RHom}_\Gamma(p,M))\\ = &\ \operatorname{Hom}_\Gamma(p,M)\end{aligned}$$

Of course we also have

$$\begin{aligned}\operatorname{Hom}_\Delta(M,p) &= H^0(\operatorname{RHom}_\Gamma(M,p))\\ &= H^0(\operatorname{RHom}_X(\mathcal{I}[1],\mathcal{P}))=0\end{aligned}$$

In case A is elliptic and σ has infinite order then these properties actually characterize normalized line bundles.

Theorem 4.1. *Let A be an elliptic cubic Artin-Schelter algebra where σ has infinite order. Let $(n_e,n_o)\in N\setminus\{(0,0)\}$. Then there is an equivalence of categories*

$$\begin{tikzcd}\mathcal{R}_{(n_e,n_o)}(X) \arrow[r, shift left, "\operatorname{Ext}^1_X(\mathcal{E},-)"] & \mathcal{C}_{(n_e,n_o)}(\Gamma) \arrow[l, shift left, "\operatorname{Tor}^\Gamma_1(-,\mathcal{E})"]\end{tikzcd}$$

where

$$\begin{aligned}\mathcal{C}_{(n_e,n_o)}(\Gamma)=\{M\in\operatorname{mod}(\Gamma)\mid \underline{\dim}M=(n_o,n_e,n_o,n_e-1)\ \textit{and}\\ \operatorname{Hom}_\Gamma(M,p)=0,\operatorname{Hom}_\Gamma(p,M)=0\ \textit{for all}\ p\in E\}.\end{aligned}$$

Although the category $\mathcal{C}_{(n_e,n_o)}$ has a fairly elementary description, it is not so easy to handle.

At this point we pick up another idea of Le Bruyn. For $\mathcal{I}\in\mathcal{R}_{(n_e,n_o)}(X)$ one has by same reasoning

$$\operatorname{RHom}_X(\mathcal{E},\mathcal{I}(-1))=M'[-1]$$

where $M'=\operatorname{Ext}^1_X(\mathcal{E},\mathcal{I}(-1))$ is the representation of Γ

$$\begin{tikzcd}H^1(X,\mathcal{I}(-4)) \arrow[r, shift left, "'X"] \arrow[r, shift right, "'Y"'] & H^1(X,\mathcal{I}(-3)) \arrow[r, shift left, "X"] \arrow[r, shift right, "Y"'] & H^1(X,\mathcal{I}(-2)) \arrow[r, shift left, "X'"] \arrow[r, shift right, "Y'"'] & H^1(X,\mathcal{I}(-1))\end{tikzcd}$$

with dimension vector (n_e-1,n_o,n_e,n_o). So it is intuitively clear that $\mathcal{I}$ is actually determined by the "middle part" i.e. the representation M^0

$$\begin{tikzcd}H^1(X,\mathcal{I}(-3)) \arrow[r, shift left, "X"] \arrow[r, shift right, "Y"'] & H^1(X,\mathcal{I}(-2)) \arrow[r, shift left, "X'"] \arrow[r, shift right, "Y'"'] & H^1(X,\mathcal{I}(-1))\end{tikzcd}$$

with dimension vector $\alpha:=\underline{\dim}M^0=(n_o,n_e,n_o)$ of the full subquiver Γ^0

$$\begin{tikzcd}-3 \arrow[r, shift left, "X_{-3}"] \arrow[r, shift right, "Y_{-3}"'] & -2 \arrow[r, shift left, "X_{-2}"] \arrow[r, shift right, "Y_{-2}"'] & -1\end{tikzcd}$$

of Γ, without relations. We now see which properties characterize M^0.

First, as M^0 is a restriction of M this manifests in a certain rank condition involving the matrices X, X', Y, Y'. For example, if A is of type A then by (4.3)

$$\operatorname{rank}\begin{pmatrix} aY'Y + cX'X & bX'Y + aY'X \\ bY'X + aX'Y & aX'X + cY'Y \end{pmatrix} \leq \dim\ker\begin{pmatrix} X'' & Y'' \end{pmatrix} = 2n_o - (n_e - 1)$$

A second important fact is that the representation M^0 is θ-stable for $\theta = (-1, 0, 1)$, shown as follows. For each $\mathcal{I} \in \mathcal{R}(n_e, n_o)(X)$ we may find an element of degree two $v \in A_2$ such that $\operatorname{Hom}_X(\mathcal{I}, \pi(A/vA)) = 0$. Through the derived equivalence (4.1) the module A/vA corresponds to a certain representation Q^0 of Γ^0 and chasing homology we get $\operatorname{Hom}_{\Gamma^0}(M^0, Q^0) = 0$. Using the Euler form on Γ^0 we also find $\operatorname{Ext}^1_{\Gamma^0}(M^0, Q^0) = 0$ hence $M^0 \perp Q^0$. This means M^0 is θ-semistable. We then show any filtration of M^0 with stable quotients has length one i.e. M^0 is θ-stable.

These two properties of M^0 fully determine M^0 and M.

Theorem 4.2. *Let A be an elliptic cubic Artin-Schelter algebra where σ has infinite order. Let $(n_e, n_o) \in N \setminus \{(0,0),(1,1)\}$. Then there is an equivalence of categories*

$$\begin{tikzcd} \mathcal{C}_{(n_e,n_o)}(\Gamma) \arrow[r, shift left, "\mathrm{Res}"] & \mathcal{D}_{(n_e,n_o)}(\Gamma^0) \arrow[l, shift left, "\mathrm{Ind}"] \end{tikzcd}$$

where

$$\mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \{F \in \operatorname{mod}(\Gamma^0) \mid \underline{\dim} F = (n_o, n_e, n_o), F \text{ is } \theta\text{-stable}, \\ \dim_k(\operatorname{Ind} F)_0 \geq n_e - 1\}.$$

Since all representations in $\mathcal{D}_{(n_e,n_o)}(\Gamma^0)$ are stable, all $\mathrm{Gl}_\alpha(k)$-orbits are closed so by putting

$$D_{(n_e,n_o)} = \{F \in \operatorname{Rep}_\alpha(\Gamma^0) \mid F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)\} /\!\!/ \mathrm{Gl}_\alpha(k)$$

we see $D_{(n_e,n_o)}$ is really the orbit space for the $\mathrm{Gl}_\alpha(k)$ action. The rank condition is a closed condition while stability is an open one, hence $D_{(n_e,n_o)}$ are locally closed varieties. Smoothness follows from the fact that the dimension of the tangent space at $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ is constant.

5 Generic type A

To complete our sketch of the proof of Theorem 1, assume A is of generic type A. Thus in the geometric data (E, σ, j), $j : E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ where E is a smooth elliptic curve and σ is given by a translation. In this case we may prove that $\coprod_{(n_e,n_o)} \mathcal{R}_{(n_e,n_o)}(X)$ is equivalent with the full subcategory of $\operatorname{coh}(X)$ with objects

$$\{\mathcal{M} \in \operatorname{coh}(X) \mid u^*\mathcal{M} \text{ is a line bundle on } E \text{ of degree zero}\}.$$

By picking a suitable line bundle $\mathcal{V}$ of degree zero on E (with different first Chern class then the appearing line bundles on E above) we have $\mathrm{RHom}_E(Li^*\mathcal{I}, \mathcal{V}) = 0$. Chasing through the derived equivalence we find a representation V of Γ^0 for which $M^0 \perp V$ for all $M^0 \in \mathcal{D}_{(n_e,n_o)}(X)$. This leads to the alternative description

$$\begin{aligned} D_{(n_e,n_o)} = \{F = (X, Y, X', Y') \in M_{n_e\times n_o}(k)^2 \times M_{n_o\times n_e}(k)^2 \mid F \perp V, \\ \mathrm{rank} \begin{pmatrix} aY'Y + cX'X & bX'Y + aY'X \\ bY'X + aX'Y & aX'X + cY'Y \end{pmatrix} \leq 2n_o - (n_e - 1)\}/\,\mathrm{Gl}_\alpha(k) \end{aligned}$$

As $\{F \in \mathrm{Rep}_\alpha(\Gamma^0) \mid F \perp V\}$ is affine we deduce $D_{(n_e,n_o)}$ is also affine.

6 The enveloping algebra

We now consider the case where $A = H_c$ is the enveloping algebra of the Heisenberg-Lie algebra

$$\begin{aligned} H_c &= k\langle x, y, z\rangle/(yz - zy, xz - zx, xy - yx - z) \\ &= k\langle x, y\rangle/(y^2x - 2yxy + xy^2, x^2y - 2xyx + yx^2). \end{aligned}$$

In the geometric data (E, σ, j) the divisor E is the double diagonal $2D$ on $\mathbb{P}^1 \times \mathbb{P}^1$. However, it is more convenient to work with the reduced divisor $E_{\mathrm{red}} = D$. Again we find that the category $\mathcal{R}(X) = \coprod_{(n_e,n_o)} \mathcal{R}_{(n_e,n_o)}(X)$ is equivalent to the full subcategory of $\mathrm{coh}(X)$ with objects

$$\begin{aligned} &\{\mathcal{M} \in \mathrm{coh}(X) \mid u^*\mathcal{M} \text{ is a line bundle on } D \text{ of degree zero}\} \\ =&\{\mathcal{M} \in \mathrm{coh}(X) \mid u^*\mathcal{M} \cong \mathcal{O}_D\} \end{aligned}$$

This means that the set $R(H_c)$ (thus also $R(A_1^\varphi)$) is bijective to the set of isoclasses

$$\{\mathcal{M} \in \mathrm{coh}(X) \mid u^*\mathcal{M} \cong \mathcal{O}_D\}/\mathrm{iso}$$

On the other hand, for a normalized line bundle $\mathcal{I}$ on X we get by adjointness

$$\mathrm{RHom}_X(\mathcal{I}, \pi(A/zA)(-1)) \cong \mathrm{RHom}_X(Li^*\mathcal{I}, \mathcal{O}_{\Delta(-1)})$$

thus $\mathrm{Hom}_X(\mathcal{I}, \pi(A/zA)) = 0$. The object $\pi(A/zA) \in \mathrm{coh}(X)$ corresponds to a representation V of Γ^0 and using the derived equivalence (4.1) once again we now have the alternative description

$$\begin{aligned} D_{(n_e,n_o)} = \{F = (X, Y, X', Y') \in M_{n_e\times n_o}(k)^2 \times M_{n_o\times n_e}(k)^2 \mid F \perp V, \\ \mathrm{rank} \begin{pmatrix} Y'Y & X'Y - 2Y'X \\ Y'X - 2X'Y & X'X \end{pmatrix} \leq 2n_o - (n_e - 1)\}/\,\mathrm{Gl}_\alpha(k) \end{aligned}$$

The condition $F \perp Q$ translates into $Y'X - X'Y$ is an isomorphism. Putting

$$\begin{cases} \mathbb{X} = X \\ \mathbb{Y} = Y \end{cases} \text{and} \begin{cases} \mathbb{X}' = (Y'X - X'Y)^{-1}X' \\ \mathbb{Y}' = (Y'X - X'Y)^{-1}Y' \end{cases}$$

one easily deduces $\mathbb{Y}\mathbb{X}' - \mathbb{X}'\mathbb{Y}' = \mathbb{I}$ and $\mathrm{rank}(\mathbb{Y}\mathbb{X}' - \mathbb{X}\mathbb{Y}' - \mathbb{I}) \leq 1$. Hence Theorem 2 follows.

7 Hilbert series of graded right ideals

In this final part we wish to describe the possible Hilbert series of reflexive graded right ideals of cubic Artin-Schelter algebras.

Let us first consider the commutative polynomial ring in three variables $k[x, y, z]$. A characterization of all possible Hilbert functions of graded ideals in $k[x, y, z]$ was given by Macaulay in 1927. To describe them, assume I is a graded right ideal of $k[x, y, z]$. There might be a lot of other ideals J which are "very closely related" to I, namely those J of the form

$$0 \to I \to J \to F \to 0$$

where F is a finite dimensional graded right A-module. To avoid this we shall assume I has projective dimension one. This means $\operatorname{Ext}^1_A(F, I) = 0$ for all finite dimensional $F \in \operatorname{grmod}(A)$.

Similar as in Section 3 it is intuitively clear that the difference function $m \mapsto \dim_k A_m - \dim_k I_m$ is linear in m for $m \gg 0$. One may show there is an (unique) integer d for which

$$\dim_k A_m - \dim_k I(d)_m = n \text{ for } m \gg 0$$

for some positive integer n. In terms of formal power series this means

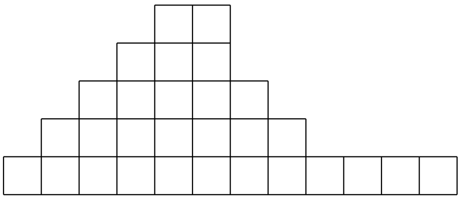
$$\begin{aligned} h_{I(d)}(t) &= h_{k[x,y,z]}(t) - \frac{s(t)}{1-t} \\ &= \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \end{aligned}$$

for some Laurent polynomial $s(t) \in \mathbb{Z}[t, t^{-1}]$ with $s(1) = n$. It turns out that $s(t)$ is a so-called *Castelnuovo polynomial* which by definition has the form

$$\begin{gathered} s(t) = 1 + 2t + 3t^2 + \cdots + ut^{u-1} + s_u t^u + \cdots + s_v t^v \\ u \geq s_u \geq \cdots \geq s_v \geq 0 \end{gathered}$$

for some integers $u, v \geq 0$. Such polynomials are visualized using the graph of a staircase function. For example, the diagram for the Castelnuovo polynomial

$$s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$$



The total number of cases is $s(1)$, called the *weight* of $s(t)$.

It is known that a formal power series $h(t) \in \mathbb{Z}((t))$ is of the form $h_{I(d)}(t)$ for some graded ideal I of projective dimension one if and only if $h(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$ for some Castelnuovo polynomial $s(t)$.

In 2004, Van den Bergh and the author showed we may generalize this result to quadratic Artin-Schelter regular algebras.

Theorem C. *Let A be a quadratic Artin-Schelter algebra. Let $s(t) \in \mathbb{Z}[t, t^{-1}]$. Then*

$$\frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

is the Hilbert series $h_{I(d)}(t)$ for some graded right A-ideal I of projective dimension one if and only if $s(t)$ is a Castelnuovo polynomial.

If in addition A is elliptic and σ has infinite order then I may be chosen reflexive.

We will now look for an analogue of Theorem C for cubic Artin-Schelter algebras A. So let I be a graded right A-ideal of projective dimension one. As indicated in Section 3 there is an (unique) integer d for which

$$\dim_k A_m - \dim_k I(d)_m = \begin{cases} n_e & \text{if } m \text{ is even,} \\ n_o & \text{if } m \text{ is odd} \end{cases} \quad \text{for } m \gg 0.$$

for some positive integers n_e, n_o. In terms of formal power series this means

$$\begin{aligned} h_I(t) &= h_A(t) - \frac{s(t)}{1-t^2} \\ &= \frac{1}{(1-t)^2(1-t^2)} - \frac{s(t)}{1-t^2} \end{aligned}$$

for some Laurent polynomial $s(t) = \sum_i s_i t^i \in \mathbb{Z}[t, t^{-1}]$ with $\sum_i s_{2i} = n_e$ and $\sum_i s_{2i+1} = n_o$.

Same techniques as for quadratic Artin-Schelter algebras reveal

Theorem 3. *Let A be a cubic Artin-Schelter algebra. Let $s(t) \in \mathbb{Z}[t, t^{-1}]$. Then*

$$\frac{1}{(1-t)^2(1-t^2)} - \frac{s(t)}{1-t^2}$$

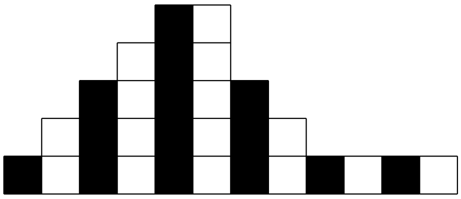
is the Hilbert series $h_{I(d)}(t)$ for some graded right A-ideal I of projective dimension one if and only if $s(t)$ is a Castelnuovo polynomial.

If in addition A is alliptic and σ has infinite order then I may be chosen reflexive.

For a Castelnuovo polynomial $s(t)$ we refer to $\sum_i s_{2i}$ as the *even weight* and $\sum_i s_{2i+1}$ as the *odd weight* of $s(t)$. For example, for

$$s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$$

we have even weight 14 (number of ■) and odd weight 15 (number of □)



Let us assume A is elliptic and σ has infinite order. For any integers n_e, n_o Theorem 3 yields

$$\mathcal{R}_{(n_e,n_o)} \neq \emptyset \Leftrightarrow \text{ there exists a Castelnuovo polynomial} \\ \text{of even weight } n_e \text{ and odd weight } n_o$$

However recall from Section 3

$$\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X) \Rightarrow \dim_k \operatorname{Ext}^1_X(\mathcal{I},\mathcal{I}) = 2(n_e - (n_e - n_o)^2)$$

whence

$$\mathcal{R}_{(n_e,n_o)}(X) \neq \emptyset \Rightarrow n_e - (n_e - n_o)^2 \geq 0$$

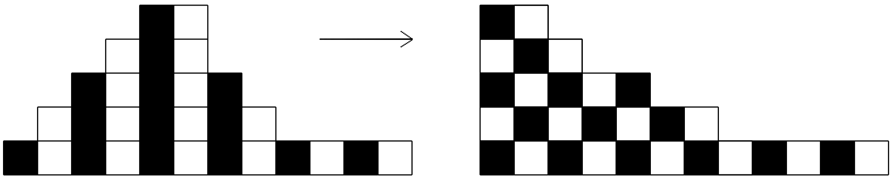
Therefore if there is a Castelnuovo polynomial of even weight n_e and odd weight n_o then[4] $n_e - (n_e - n_o)^2 \geq 0$. This inequality was clearly a hint for us, and the converse was easy to prove by construction. Therefore

$$\mathcal{R}_{(n_e,n_o)}(X) \neq \emptyset \Leftrightarrow (n_e, n_o) \in N = \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$$

We would like to end with a combinatorical by-product. By shifting the rows in a Castelnuovo diagram in such a way they are left aligned one sees a bijective correspondence between Castelnuovo diagrams and partitions in distinct parts. The configuration in black and white cases then form a chessboard formation on the Young diagram. For example the Castelnuovo polynomial

$$s(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 5t^5 + 3t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11}$$

corresponds to the partition $(12, 7, 5, 3, 2)$



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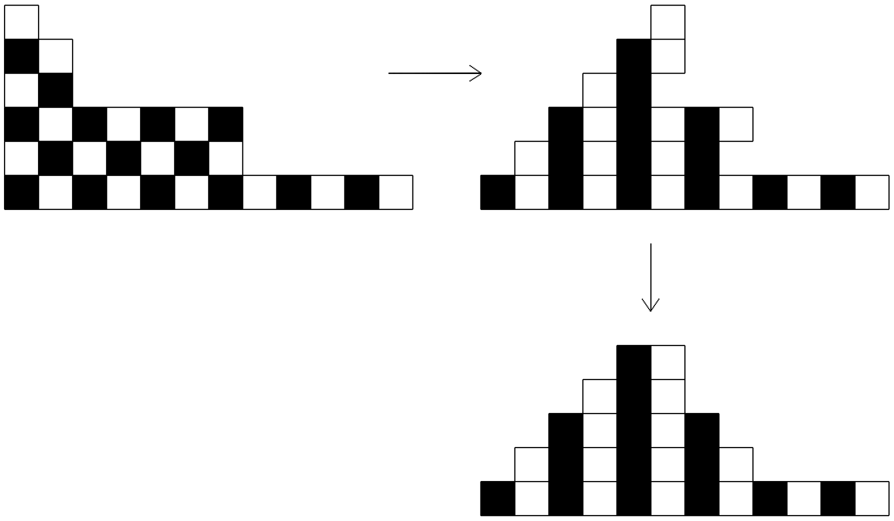
[4] We also found a direct (combinatorical) proof of this.

This procedure is reversible for all partitions (but not bijective), for example the partition $(12, 7, 7, 2, 2, 1)$ also corresponds to the above Catelnuovo polynomial.



To set some notation, for any partition λ we let $b(\lambda)$ resp. $w(\lambda)$ denote the number of black resp. white cases in the draughts colouring of the Young diagram of λ. We obtain

Theorem 7.1. *Given positive integers n_e, n_o there exists a partition λ with $b(\lambda) = n_e$ and $w(\lambda) = n_o$ if and only if $n_e - (n_e - n_o)^2 \geq 0$.*

It is not surprising this combinatorical result was already known, however it seems to be difficult to track the complete history of it. Anthony Henderson pointed out to us that Theorem 7.1 is a special case of Chung's conjecture, proved by Robinson in 1952 by means of representation theory of the symmetric group. Using that theory we even have information of the number of possible Hilbert series

$$\begin{aligned} & \text{the number of possible Hilbert series for ideals} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{of even weight } n_e \text{ and odd weight } n_o \\ = \; & \text{the number of partitions } \lambda \text{ in odd parts with } b(\lambda) = n_e \text{ and } w(\lambda) = n_o \\ = \; & \text{the number of partitions of } n_e - (n_e - n_o)^2. \end{aligned}$$

An elementary proof of this last fact (avoiding representation theory) is unknown to us, and perhaps even unlikely.