Ideal classes of three dimensional Sklyanin algebras

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This talk is based on joint work with Michel Van den Bergh.

1 Introduction

Consider the first Weyl algebra

$$A_1 = \mathbb{C}\langle x, y\rangle/(yx - xy - 1)$$

There is a classification of its right ideals.

Theorem 1.1. (Cannings and Holland, Wilson) [1] *Let $\mathcal{R}$ be the set of isomorphism classes of right A_1-ideals. $G = \mathrm{Aut}(A_1)$ has a natural action on $\mathcal{R}$, where*

- *the orbits of the G-action on $\mathcal{R}$ are indexed* [2] *by $\mathbb{N}$*
- *The orbit corresponding to $n \in \mathbb{N}$ is in natural bijection with the n'th Calogero-Moser space*

$$C_n = \{X, Y \in M_n(\mathbb{C}) \mid \mathrm{rk}(YX - XY - \mathrm{id}) = 1\}/\,\mathrm{Gl}_n(\mathbb{C})$$

where $\mathrm{Gl}_n(\mathbb{C})$ acts by simultaneous conjugation on (X, Y).

Berest and Wilson gave a new proof of this theorem using noncommutative algebraic geometry. That such an approach should be possible was in fact anticipated very early by Le Bruyn who already came very close to proving the above theorem.

Let me indicate how the methods of noncommutative algebraic geometry may be used to prove Theorem 1.1. Introduce the *homogenized Weyl algebra*

$$H = k\langle x, y, z\rangle/(zx - xz, zy - yz, yx - xy - z^2)$$

[1] First proved by Cannings and Holland, using a description of $\mathcal{R}$ in terms of adelic Grassmanian. Wilson established a relation between the adelic Grassmannian and C_n.

[2] The fact that $\mathcal{R}/G \cong \mathbb{N}$ has also been proved by Kouakou in his (unpublished) PhD-thesis.

we have that $A_1 = H/(z-1)$ and A_1-ideals correspond to reflexive graded right ideals of H. Now H defines a noncommutative projective plane $\mathbb{P}^2_q$ (in the sense of Artin and Zhang). Describing $\mathcal{R}$ then becomes equivalent to describing certain objects on $\mathbb{P}^2_q$. Objects on $\mathbb{P}^2_q$ have finite dimensional cohomology groups and these may be used to define moduli spaces, just as in the ordinary commutative case.

We start with the observation that there are many more algebras defining a noncommutative plane, and in some sense the generic ones which have "nice" properties (so-called Artin-Schelter regular algebras of type A) are the three dimensional Sklyanin algebras

$$\mathrm{Skl}_3(a,b,c) = k\langle x,y,z\rangle/(ayz+bzy+cx^2, azx+bxz+cy^2, axy+byx+cz^2)$$

where $(a,b,c) \in \mathbb{P}^2 \setminus F$, for a (known) finite set F. The Hilbert series of $\mathrm{Skl}_3(a,b,c)$ is the same as the Hilbert series of the polynomial ring $k[x,y,z]$, namely $(1-t)^{-3}$, and $\mathrm{Skl}_3(a,b,c)$ has a central element of degree three.
Let $\mathcal{R}$ be the set of reflexive graded $\mathrm{Skl}_3(a,b,c)$-ideals, considered up to isomorphism and shift of grading. We obtained the following result

Theorem 1.2. (Van den Bergh and De Naeghel) *There exist smooth affine connected varieties D_n of dimension $2n$ such that $\mathcal{R}$ is naturally in bijection with $\coprod_n D_n$.*

We would like to think of the D_n as elliptic Calogero-Moser spaces. We have that D_0 is a point and D_1 is the complement of E under a natural embedding in $\mathbb{P}^2$.

Remark 1.3. Nevins and Stafford announced a similar theorem, they work in a more general setting where the translation of the elliptic curve may have finite order. Although they did not prove their varieties are affine.

The aim is to give an idea how we obtain the varieties D_n.

2 Noncommutative projective planes

For the rest of this talk, $A = \mathrm{Skl}_3(a,b,c)$ will be a three dimensional Sklyanin algebra. It is an Artin-Schelter regular algebra of global dimension three and therefore determined (as shown by Artin,Tate and Van den Bergh) by geometric data $(E,\sigma,\mathcal{L})$ where

- $E \overset{j}{\hookrightarrow} \mathbb{P}^2$ is an elliptic curve,
- $\sigma \in \mathrm{Aut}(E)$ and
- $\mathcal{L}$ is a line bundle on E.

Since A is a Sklyanin algebra, E is smooth, σ is a translation on E and $\mathcal{L} = j^*\mathcal{O}_{\mathbb{P}^2}(1)$. We will also assume that σ has infinite order.
Let

$$\mathrm{Tails}(A) = \mathrm{GrMod}(A)/\operatorname{Tors}(A)$$

where $\mathrm{GrMod}(A)$ is the category of graded right A-modules and $\mathrm{Tors}(A)$ its full subcategory of modules which are the sum of their finite dimensional submodules. $\mathrm{tails}(A)$ is the full subcategory of noetherian objects. Denote by $\pi : \mathrm{GrMod}(A) \to \mathrm{Tails}(A)$ the exact quotient functor. We denote $\pi A = \mathcal{O}$. Objects in $\mathrm{Tails}(A)$ will be written with script letters. The shift of grading in $\mathrm{GrMod}(A)$ induces an automorphism $\mathrm{sh} : \mathcal{M} \to \mathcal{M}(1)$ on $\mathrm{Tails}(A)$ which we call the shift functor. Following Artin and Zhang, we define the projective scheme

$$\mathbb{P}^2_{\mathrm{q}} = \mathrm{Proj}\, A := (\mathrm{tails}(A), \mathcal{O}, \mathrm{sh})$$

and put

$$\begin{aligned} \mathrm{Qcoh}(\mathbb{P}^2_{\mathrm{q}}) &:= \mathrm{Tails}(A) \\ \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}}) &:= \mathrm{tails}(A) \end{aligned}$$

and think of them as the (quasi)coherent sheaves on $\mathbb{P}^2_{\mathrm{q}}$, even though they are not really sheaves.

Let $\mathcal{R}$ be the set of reflexive graded A-ideals, considered up to isomorphism and shift of grading ($M \in \mathrm{GrMod}(A)$ reflexive means that $M^{**} = M$ where $M^* = \underline{\mathrm{Hom}}_A(M, A)$ is the graded dual of M).
Images of reflexive A-ideals under π will be called *line bundles* (since they behave like vector bundles of rank one). If M is an A-ideal we may consider its image $\mathcal{M} = \pi M$ in $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$. We may consider the cohomology groups of $\mathcal{M}$

$$H^i(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M}) := \mathrm{Ext}^i_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{O}, \mathcal{M}).$$

3 Find a good shift l such that we get track about the cohomology groups of $\mathcal{M}(l)$

To do this consider the Grothendieck group K_0 of $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$. We have an isomorphism of additive groups (as shown by Mori and Smith)

$$\theta : K_0 \to \mathbb{Z}[t, t^{-1}] \,/ (1-t)^3 \;:\; [\pi N] \mapsto \overline{h_N(t)(1-t)^3}$$

where $h_n(t) = \sum_i \dim_k N_i t^i$ is the Hilbert series of $N \in \mathrm{grmod}(A)$. Using $h_{N(1)}(t) = t^{-1} h_N(t)$ it is clear that $\{[\mathcal{O}], [\mathcal{O}(-1)], [\mathcal{O}(-2)]\}$ is a $\mathbb{Z}$-module basis for K_0 (the images under θ are $\overline{1}, \overline{t}, \overline{t^2}$) but we will use a more natural basis using line modules and point modules over A.

- A *point* module $P \in \mathrm{grmod}\, A$ is a module such that
  - $P_0 = k$
  - P_0 generates P
  - $h_P(t) = (1-t)^{-1}$.

It turns out that (Artin, Tate and Van den Bergh) there is a bijection between point modules over A and the closed points of E.

- A *line* module $S \in \operatorname{grmod} A$ is a module such that
  - $S_0 = k$
  - S_0 generates L
  - $h_S(t) = (1-t)^{-2}$.

  Line modules over A are of the form $S = A/uA$ where $u \in A_1$ is a homogeneous element of degree one.

Fix some point module P and a line module S over A. The images of $[\mathcal{O}], [\mathcal{S}], [\mathcal{P}]$ are resp. $\overline{1}, \overline{1-t}, \overline{(1-t)^2}$ so they form a basis of K_0. For $M \in \mathcal{R}$ we have

$$[\mathcal{M}] = [\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{P}]$$

for some integers a, b (the coefficient of $[\mathcal{O}]$ is the rank of M, so it is one since M is an ideal).
The idea is to pick a suitable shift l such that $[\mathcal{M}(l)]$ takes a simple form expressed in terms of our basis, hoping to give some information about the cohomology of $\mathcal{M}(l)$. So we only need to know the shifts of our basis elements. Using θ it is easy to see that

$$\begin{aligned}[\mathcal{O}(1)] &= [\mathcal{O}] + [\mathcal{S}] + [\mathcal{P}] \\ [\mathcal{S}(1)] &= [\mathcal{S}] + [\mathcal{P}] \\ [\mathcal{P}(1)] &= [\mathcal{P}]\end{aligned}$$

and if we shift M by $-a$ we get

$$[\mathcal{M}(-a)] = [\mathcal{O}] - n[\mathcal{P}]$$

where $n = a(a+1)/2 - b$. The integer n attached to M is called the *invariant* of M, and say that the line bundle $\mathcal{M}(l)$ is *normalized.*
The following theorem gives information about the cohomology of normalized line bundles.

Theorem 3.1. *Let $\mathcal{M}$ be a normalized line bundle, $[\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]$. If $n = 0$ then $\mathcal{M} \cong \mathcal{O}$. If $n \neq 0$ then*

1. $H^0(\mathbb{P}^2_q, \mathcal{M}(l)) = 0$ *for* $l \leq 0$,
   $H^2(\mathbb{P}^2_q, \mathcal{M}(l)) = 0$ *for* $l \geq -3$;
2. $\dim_k H^1(\mathbb{P}^2_q, \mathcal{M}) = n - 1$
   $\dim_k H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) = n$
   $\dim_k H^1(\mathbb{P}^2_q, \mathcal{M}(-2)) = n$
   $\dim_k H^1(\mathbb{P}^2_q, \mathcal{M}(-3)) = n - 1$

and as a consequence $n > 0$.

This result is similar to the one for the homogenized Weyl algebra, however in that case the computations rely on the existence of a central element in degree one (namely z). So they do not apply in a straightforward way to the case we consider since we now have a cental element of degree three.

We have a natural bijection

$$\mathcal{R} = \{\text{reflexive right } A\text{-ideals}\}/\text{ iso, shift } \leftrightarrow \{\text{ normalized line bundles on } \mathbb{P}^2_{\mathrm{q}}\ \}/\text{iso}$$

For the sequel it will be appropriate to work with a category. Let $\mathcal{R}_n$ the category in which the objects are the normalized line bundles with invariant n on $\mathbb{P}^2_{\mathrm{q}}$ and the morphisms are the isomorphisms in $\operatorname{coh}\mathbb{P}^2_{\mathrm{q}}$. Thus $\mathcal{R}_n$ is a groupoid.

4 A derived equivalence translates $\mathcal{R}_n$ to linear algebra

In the commutative case it is well-known that we have an equivalence of derived categories (Beilinson)

$$\mathrm{D}^b(\operatorname{coh}\mathbb{P}^2) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\operatorname{mod}\Delta)$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}}$ and $\operatorname{mod}\Delta$ is the category of finite dimensional representations of the quiver Δ

$$-2 \quad \begin{array}{c} \xrightarrow{X_{-2}} \\ \xrightarrow{Y_{-2}} \\ \xrightarrow{Z_{-2}} \end{array} \quad -1 \quad \begin{array}{c} \xrightarrow{X_{-1}} \\ \xrightarrow{Y_{-1}} \\ \xrightarrow{Z_{-1}} \end{array} \quad 0$$

with relations

$$\begin{cases} Y_{-2}Z_{-1} = Z_{-2}Y_{-1} \\ Z_{-2}X_{-1} = X_{-2}Z_{-1} \\ X_{-2}Y_{-1} = Y_{-2}X_{-1} \end{cases}$$

We have a similar situation for our Sklyanin algebra A. There is an equivalence of derived categories (follows from a more general theorem of Bondal)

$$\mathrm{D}^b(\operatorname{coh}\mathbb{P}^2_{\mathrm{q}}) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\operatorname{mod}\Delta)$$

where $\mathcal{E} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and Δ is the quiver

$$-2 \quad \begin{array}{c} \xrightarrow{X_{-2}} \\ \xrightarrow{Y_{-2}} \\ \xrightarrow{Z_{-2}} \end{array} \quad -1 \quad \begin{array}{c} \xrightarrow{X_{-1}} \\ \xrightarrow{Y_{-1}} \\ \xrightarrow{Z_{-1}} \end{array} \quad 0$$

with relations

$$\begin{cases} aY_{-2}Z_{-1} + bZ_{-2}Y_{-1} + cX_{-2}X_{-1} = 0 \\ aZ_{-2}X_{-1} + bX_{-2}Z_{-1} + cY_{-2}Y_{-1} = 0 \\ aX_{-2}Y_{-1} + bY_{-2}X_{-1} + cZ_{-2}Z_{-1} = 0 \end{cases}$$

We would like to understand the image of $\mathcal{R}_n$ under the equivalence. Let's see what happens to an object $\mathcal{M}$ of $\mathcal{R}_n$. Consider $\mathcal{M}$ as a complex in $\mathrm{D}^b(\mathrm{coh}\,\mathbb{P}^2_{\mathrm{q}})$ of degree zero. Due to the previous theorem, the image of this complex is concentrated in degree one

$$\mathrm{RHom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{E}, \mathcal{M}) = M[-1]$$

where $M = \mathrm{Ext}^1(\mathcal{E}, \mathcal{M})$. Hence it is a representation of Δ. How do we build the linear maps? By functoriality, multiplication by $x \in A$ induces linear maps

$$H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M}(-2)) \quad \overset{M(X_{-2})}{\longrightarrow} \quad H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M}(-1)) \quad \overset{M(X_{-1})}{\longrightarrow} \quad H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M})$$

and similar for multiplication with y, z hence M is determined by the following representation of Δ

$$H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M}(-2)) \quad \begin{matrix} \overset{M(X_{-2})}{\longrightarrow} \\ \overset{M(Y_{-2})}{\longrightarrow} \\ \overset{M(Z_{-2})}{\longrightarrow} \end{matrix} \quad H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M}(-1)) \quad \begin{matrix} \overset{M(X_{-1})}{\longrightarrow} \\ \overset{M(Y_{-1})}{\longrightarrow} \\ \overset{M(Z_{-1})}{\longrightarrow} \end{matrix} \quad H^1(\mathbb{P}^2_{\mathrm{q}}, \mathcal{M})$$

¿From the previous theorem it is clear that $\dim M = (n, n, n-1)$. The next thing to do is see how the reflexivity of $\mathcal{M}$ is translated through the derived equivalence. Consider a point module P over A, $\mathcal{P} = \pi P$. $\mathcal{M}$ reflexive means that $\mathrm{Ext}^1(\mathcal{P}, \mathcal{M}) = 0$. On the other hand,

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{P}, \mathcal{M}) &= H^0(\mathrm{RHom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{P}, \mathcal{M}[1])) \\ &= H^0(\mathrm{RHom}_{\Delta}(p, M)) \\ &= \mathrm{Hom}_{\Delta}(p, M) \end{aligned}$$

Of course we also have

$$\begin{aligned} \mathrm{Hom}_{\Delta}(M, p) &= H^0(\mathrm{RHom}_{\Delta}(M, p)) \\ &= H^0(\mathrm{RHom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{M}[1], \mathcal{P})) = 0 \end{aligned}$$

Actually these properties characterise the normalized line bundles.

Theorem 4.1. *Let $n \geq 1$. There is an equivalence of categories*

$$\mathcal{R}_n \underset{\mathrm{Tor}_1^{\Delta}(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E},-)}{\rightleftarrows}} \mathcal{C}_n$$

where

$$\mathcal{C}_n = \{M \in \mathrm{mod}\,\Delta \mid \underline{\dim} M = (n,n,n-1) \textit{ and } \mathrm{Hom}_\Delta(M,p) = 0, \mathrm{Hom}_\Delta(p,M) = 0 \textit{ for all } p \in E\}.$$

Although the category $\mathcal{C}_n$ has a fairly elementary description, it is not so easy to handle. We will now give another description of $\mathcal{R}$.

5 The varieties D_n

Let $\mathcal{M} \in \mathcal{R}_n$ a normalized line bundle, $n \neq 0$. As we noticed in the previous section, $\mathcal{M}$ is determined by a representation M of the quiver Δ, because $\mathrm{Ext}^i(\mathcal{E},\mathcal{M}) = 0$ for $i \neq 1$.
As Le Bruyn observed in the Weyl algebra case, $\mathcal{M}(-1)$ is also determined by such a representation because $\mathrm{Ext}^i(\mathcal{E},\mathcal{M}(-1)) = 0$ for $i \neq 1$. By an argument of Baer we get that $\mathcal{M}$ is actually determined by a Kronecker module M^0 of the quiver Δ^0, the full subquiver of Δ consisting of the vertices $-2,-1$

$$-2 \quad \begin{matrix} \xrightarrow{X_{-2}} \\ \xrightarrow{Y_{-2}} \\ \xrightarrow{Z_{-2}} \end{matrix} \quad -1$$

It is worth to be more precise here. Let $\mathrm{Res} : \mathrm{Mod}\,\Delta \to \mathrm{Mod}\,\Delta^0$ be the obvious restriction functor. So $M^0 = \mathrm{Res}\,M$. Res has a left adjoint which we denote by Ind. Note that $\mathrm{Res} \circ \mathrm{Ind} = \mathrm{id}$. We have

Lemma 5.1. *If $\mathcal{M} \in \mathcal{R}_n$, $n \neq 0$ and $M = \mathrm{Ext}^1(\mathcal{E},\mathcal{M})$ then $M = \mathrm{Ind}\,\mathrm{Res}\,M$.*

Let me formulate the remaining results in a theorem.

Theorem 5.2. *Let $n \geq 1$. There exists a $V \in \mathrm{mod}(\Delta^0)$ such that*

1. *for all $M \in \mathcal{C}_n$ we have $\mathrm{Res}\,M \perp V$.*
   As a consequence[3]*, $\mathrm{Res}\,M$ is θ-semistable for $\theta = (-1,1)$.*

2. *The functors Res and Ind define inverse equivalences between $\mathcal{C}_n$ and the following category*

$$\mathcal{D}_n = \{F \in \mathrm{mod}(\Delta^0) \mid \underline{\dim} F = (n,n), F \perp V, \dim(\mathrm{Ind}\,F)_0 \geq n-1\}$$

[3]Due to a more general theory of Schofield and Van den Bergh.

3. *The representations in $\mathcal{D}_n$ are θ-stable.*

4. *Let $\alpha = (n, n)$. The affine variety*

$$D_n = \{F \in \mathrm{Rep}(\Delta^0, \alpha) \mid F \in \mathcal{D}_n\}/\,\mathrm{Gl}(\alpha)$$

is smooth and connected[4] *of dimension $2n$ and the isomorphism classes in $\mathcal{R}_n$) are in natural bijection with the points in D_n.*

Let us try to make the analogy with the Weyl algebra case and the commutative situation. We have the following alternative description of the elliptic Calogero-Moser spaces D_n (where $n \geq 1$)

$$\mathrm{D}_n = \{F \in \mathrm{Rep}(\Delta^0, \alpha) \mid F \text{ stable and } \operatorname{rk} Q_F = 2n+1)\}/\,\mathrm{Gl}(\alpha)$$

where the 3×3 matrix Q reflects the defining equations of our Sklyanin algebra

$$\begin{pmatrix} ayz + bzy + cx^2 \\ azx + bxz + cy^2 \\ axy + byx + cz^2 \end{pmatrix} = Q \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ where } Q = \begin{pmatrix} cx & bz & ay \\ az & cy & bx \\ by & ax & cz \end{pmatrix}$$

and Q_F is the $3n \times 3n$ matrix over k obtained from Q by replacing x, y, z by $F(X_{-2}), F(Y_{-2}), F(Z_{-2})$.
For the ordinary Calogero-Moser spaces C_n, reformulating[5] Theorem 1.1 we obtain

$$C_n = \{F \in \mathrm{Rep}(\Delta^0, \alpha) \mid F \text{ stable and } \operatorname{rk} W_F = 2n+1)\}/\,\mathrm{Gl}(\alpha)$$

where

$$W = \begin{pmatrix} -y & x & z \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$$

Finally, for the commutative case, it was shown by Barth and Hulek[6] that (for $n \geq 2$)

$$M(2,0,n) = \{F \in \mathrm{Rep}(\Delta^0, \alpha) \mid F \text{ stable and } \operatorname{rk} A_F = 2n+2\}/\,\mathrm{Gl}(\alpha)$$

where $M(2,0,n)$ is the moduli space of stable rank 2 vector bundles on $\mathbb{P}^2$ with first Chern class zero and second Chern class n, and

$$A = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

[4]In fact our proof uses the description of the Hilbert series of reflexive graded right A-ideals which occur, see end of the talk.
[5]Though not entirely straightforward
[6]Hulek used an equivalent definition of a stable representation, namely properly stable.

making the analogy clear.

To end this talk, observe that each point $x \in D_n$ corresponds to an isoclass $[M]$ of some graded reflexive right ideal M of A. We may consider the map

$$H : D_n \to \mathbb{Z}((t)) : x \to h_M(t)$$

sending a point to the Hilbert series of the corresponding ideal. The Hilbert series which occur are the same ones as the Hilbert series for subschemes of dimension zero and degree n on $\mathbb{P}^2$, i.e. configurations of n points on $\mathbb{P}^2$. As a byproduct, this gives us a proof for the fact that D_n is connected. So D_n is really an analogue for the Hilbert scheme of points on $\mathbb{P}^2$.
Considering subsets of points of D_n which have the same image under the map H determines a stratification of D_n. We have a good idea which strata are contained in the closure of other ones, using maps from ideals to truncated pointmodules. Moreover, these ideas may be used to get new results for the Hilbert scheme of points on $\mathbb{P}^2$. Studying these incidence problems, we really observe a difference between D_n and the Hilbert scheme of points on $\mathbb{P}^2$.