Hilbert schemes of points on quantum projective planes

talk séminaire d'algèbre
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- slides and outline talk available

Based on joined work with M. Van den Bergh.

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1. Hilbert schemes of points

1.1. Hilbert scheme of points on $\mathbb{P}^2$

- k algebraically closed field, char. zero
  $k[x, y, z]$ polynomial ring
  $\mathbb{P}^2 = \operatorname{Proj} A$ projective plane

- subschemes X of dimension zero, degree n parameterized by Hilbert scheme $\mathrm{Hilb}_n(\mathbb{P}^2)$
  X as a set: n points in $\mathbb{P}^2$.

- $\mathrm{Hilb}_n(\mathbb{P}^2)$ is smooth connected projective variety of dimension $2n$

Aim: generalize $\mathrm{Hilb}_n(\mathbb{P}^2)$ to certain non-commutative deformations of $\mathbb{P}^2$.

1.2. Quantum polynomial rings

- Fix graded k-algebra $A = k + A_1 + A_2 + \dots$ GrMod A, grmod A, Hom_A, Ext^i_A, $\underline{\mathrm{Ext}}^i_A$
- *Artin-Schelter algebra of dimension* 3 if

(i) $\operatorname{gl dim} A = 3$

(ii) A has polynomial growth

(iii) A is Gorenstein, i.e. for some $l \in \mathbb{Z}$

$$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_A k(l) & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

- *quantum polynomial ring* (qpr) if also Koszul

$$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

- qpr's are left and right noetherian domains and

$$h_A(t) = h_{k[x,y,z]}(t) = \frac{1}{(1-t)^3}$$

1.3. Examples

- Commutative polynomial ring $k[x,y,z]$ is a quantum polynomial ring.

- First Weyl algebra

$$A_1 = k\langle x,y\rangle/(yx - xy - 1)$$

Homogenize:

$$H = k\langle x,y,z\rangle/(zx - xz, zy - yz, xy - yx - z^2)$$

is a quantum polynomial ring.

- Generic qpr's: *three dim. Sklyanin algebras*

$$\mathrm{Skl}_3(a,b,c) = k\langle x,y,z\rangle/(f_1,f_2,f_3)$$

where

$$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases}$$

and $a,b,c \in k$ generic scalars.

1.4. Quantum projective planes

$$\text{Tails}\, A = \text{GrMod}\, A / \text{Tors}\, A$$

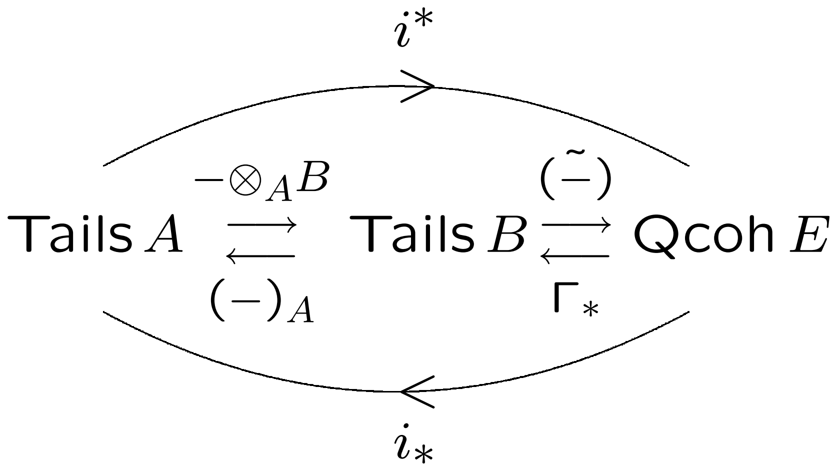
Artin and Zhang: *projective quantum plane*

$$\mathbb{P}^2_{\mathsf{q}} = \text{Proj}\, A := (\text{Tails}\, A, \mathcal{O}, \text{sh})$$

Artin, Tate and Van den Bergh:

$$A \longleftrightarrow B(E, \sigma, \mathcal{L}) \text{ where } \begin{cases} \cdot\ E \hookrightarrow \mathbb{P}^2 \text{ is } \mathbb{P}^2 \text{ (linear)} \\ \text{or divisor deg. 3 (elliptic)} \\ \cdot\ \sigma \in \text{Aut}\, E \\ \cdot\ \mathcal{L} \text{ line bundle on } E \end{cases}$$

- A linear: $B = A$
  A elliptic: $B = A/gA$ with $g \in A_3$ central
- $\text{Tails}\, B$ and $\text{Qcoh}\, E$ are equivalent



1.5. Examples

- Commutative polynomial ring $k[x, y, z]$ is linear qpr: $E = \mathbb{P}^2$, $\sigma = \text{id}$

- Homogenized Weyl algebra

  $$H = k\langle x, y, z\rangle/(zx - xz, zy - yz, xy - yx - z^2)$$

  is elliptic qpr:
  - E is given by $z^3 = 0$: "triple" line in $\mathbb{P}^2$
  - $\sigma \in \operatorname{Aut} E$ is an infinit. transl. $o(\sigma) = \infty$

- Three dimensional Sklyanin algebra

  $$\mathsf{Skl}_3(a, b, c) = k\langle x, y, z\rangle/(f_1, f_2, f_3)$$

  $$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases}$$

  is elliptic qpr:
  - E is smooth elliptic curve in $\mathbb{P}^2$
  - $\sigma \in \operatorname{Aut} E$ is translation on E

1.6. Hilbert scheme of points on $\mathbb{P}^2_q$

- $A =$ quantum polynomial ring
  $\mathbb{P}^2_q = \operatorname{Proj} A$ corresp. quantum plane
  $\mathrm{Hilb}_n(\mathbb{P}^2_q) =$?

- $\mathbb{P}^2_q$ has very few zero dimensional non-commutative subschemes

- for $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$ its graded ideal $I = I_X$ is

  1. torsionfree
  2. of projective dimension one
  3. $\dim_k A_m - \dim_k I_m = n$ for $m \gg 0$

  and $\mathrm{Hilb}_n(\mathbb{P}^2)$ parameterizes modules s.t. (1,2,3) hold.

- $\mathrm{Hilb}_n(\mathbb{P}^2_q) =$ scheme parameterizing graded right A-modules s.t. (1,2,3) hold.

Natural to consider also

$$\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})^{\mathsf{inv}} = \{I \in \mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}}) \mid I \text{ reflexive } \}$$

Theorem 1. *Let A quantum polynomial ring.*

1. $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$ *is smooth connected projective variety of dimension* $2n$*.*

2. $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})^{\mathsf{inv}}$ *is open in* $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$ *(and dense if A elliptic, $o(\sigma) = \infty$)*

3. *In case $A = \mathsf{Skl}_3(a, b, c)$, $o(\sigma) = \infty$:* $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})^{\mathsf{inv}}$ *is smooth connected affine variety of dimension* $2n$*.*

Aim second part: Sketch an intrinsic proof for the connectedness.

1.7. Examples

- Commutative polynomial ring $k[x, y, z]$:

  $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}}) = \mathsf{Hilb}_n(\mathbb{P}^2)$,
  $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})^{\mathsf{inv}} = \emptyset$ for $n > 0$.

- Homogenized Weyl algebra

  $H = k\langle x, y, z\rangle/(zx - xz, zy - yz, xy - yx - z^2)$

  $\coprod_n \mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})^{\mathsf{inv}} \cong \{$ graded right A_1-ideals $\}/$ iso

  $$I \mapsto I[z^{-1}]_0$$

  Cannings and Holland, Wilson:

  Theorem 2. $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})^{\mathsf{inv}} = C_n$ *where*

  $C_n = \{X, Y \in M_n(k) \mid \mathsf{rk}([Y, X] - \mathsf{id}) \leq 1\}/\, \mathsf{Gl}_n(k)$

  is the n-th Calogero-Moser space

2. Stratification and connectedness

Idea to prove connectedness of $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$: Let

$$\Gamma_n = \{h_I(t) \mid I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})\}$$

and define

$$\mathrm{Hilb}_h(\mathbb{P}^2_{\mathsf{q}}) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}}) \mid h_I(t) = h(t)\}$$

- Yields a stratification

  $$\mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}}) = \bigcup_{h \in \Gamma_n} \mathrm{Hilb}_h(\mathbb{P}^2_{\mathsf{q}})$$

  into smooth, non-empty connected locally closed subsets

- If $I \in \mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$:

  $$\dim \mathrm{Hilb}_{h(I)}(\mathbb{P}^2_{\mathsf{q}}) = \dim_k \mathrm{Ext}^1_A(I, I)$$

- From a formula for $\dim_k \mathrm{Ext}^1_A(I, I)$: unique stratum of max. dimension $2n$

2.1. Hilbert scheme of points on $\mathbb{P}^2$

For $X \in \mathrm{Hilb}_n(\mathbb{P}^2)$:

- graded ideal I_X,coordinate ring $A(X) = A/I_X$
- Hilbert function of X

  $$h_X : \mathbb{N} \to \mathbb{N} : d \mapsto h_X(d) := \dim\,(A(X))_d$$

  i.e. $h_X(d)$ is the rank of the evaluation function in the points of X

  $$\theta : A_d \to k^n$$

  $h_X(d) =$ number of conditions for a plane curve of degree d to contain X.

- $h_X(t) := \sum_d h_X(d)t^d = h_A(t) - h_{I_X}(t)$
- Characterization of all possible Hilbert functions of graded ideals in $k[x_1, \ldots, x_n]$ was given by Macaulay.

Example: $X =$ three points in $\mathbb{P}^2$
Clear: $h_X(0) = 1$ and $h_X(d) = n$ for $d \geq n - 1$
For small d: more complicated.

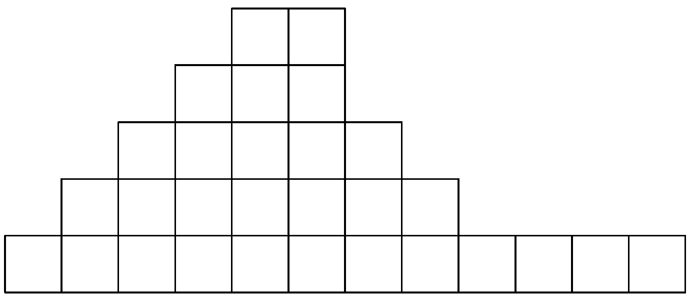
As n grows: the number of Hilbert functions increases rapidly. Castelnuovo introduced

$$s = s_X : \mathbb{N} \to \mathbb{N} : l \mapsto s_X(d) = h_X(d) - h_X(d-1)$$

which satisfies

$$\begin{cases} s(0) = 1, s(1) = 2, \ldots, s(u) = u + 1 \\ s(u) \geq s(u+1) \geq \ldots \quad \text{for some } u \geq 0 \\ s(d) = 0 \text{ for } d \gg 0 \\ \sum s(d) = n \text{ (weight of } s\text{)} \end{cases}$$

- call such functions *Castelnuovo functions*
- represented by graphs in the form of a stair



1 2 3 4 5 5 3 2 1 1 1 1

Known (Gruson and Peskine):

Theorem 3. $h_X \mapsto s_X$ *gives bijection*

$$\{h_X \mid X \in \mathsf{Hilb}_n(\mathbb{P}^2)\}$$
$$\updownarrow$$
$$\{\text{ \textit{Castelnuovo functions of weight} } n\}$$

2.1. Hilbert scheme of points on $\mathbb{P}^2_{\mathsf{q}}$

- $A =$ quantum polynomial ring, $\mathbb{P}^2_{\mathsf{q}} = \operatorname{Proj} A$ corresp. quantum plane, $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$ as before.

- If $I \in \mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$ then

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s_I(t)}{1-t}$$

  for some $s_I(t) \in \mathbb{Z}[t, t^{-1}]$.

- Let $s_I : \mathbb{Z} \to \mathbb{Z}$ s.t. $s_I(t) = \sum_d s_I(d) t^d$.

Theorem 4. *$h_I(t) \mapsto s_I$ gives bijection*

$$\Gamma_n = \{h_I(t) \mid I \in \mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})\}$$
$$\updownarrow$$
$$\{ \text{ Castelnuovo functions of weight } n\}$$

Sketch of the proof for $\uparrow$*.* Assume: A elliptic. Fix Castelnuovo function s weight n.

To prove: There exists a torsion free graded right ideal I s.t.

$$\operatorname{pd} I = 1 \text{ and } h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

Assume such an I, say with resolution

$$0 \to \oplus_i A(-i)^{b_i} \to \oplus_i A(-i)^{a_i} \to I \to 0$$

Then:

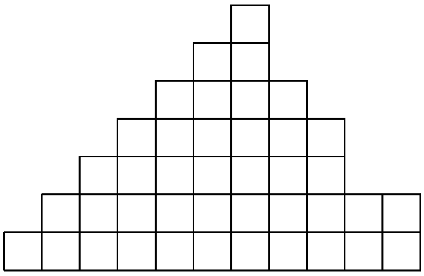
- $\sum_i (a_i - b_i) t^i = (1-t)^3 h_I(t)$
- $L_1 i^* \pi I \to \oplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{M} \oplus_i \mathcal{O}_E(-i)^{a_i} \to i^* \pi I \to 0$

If I is reflexive:

- $0 \to \oplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{M} \oplus_i \mathcal{O}_E(-i)^{a_i} \to i^* \pi I \to 0$
- $M_p = M \otimes_E \mathcal{O}_p$ has maximal rank, $\forall p \in E$.

We try to reverse this proces.

Example: s of weight 41, given by



Let H be the linear space

$$\mathrm{Hom}_E(\mathcal{O}(-8)\oplus\mathcal{O}(-10)^2\oplus\mathcal{O}(-12)^2, \mathcal{O}(-7)^3\oplus\mathcal{O}(-9)\oplus\mathcal{O}(-11)^2)$$

where $\mathcal{O} = \mathcal{O}_E$. It is sufficient to prove

$$\exists M \in H : \forall p \in E : \mathrm{rank}\, M_p = 5$$

Observe that for any $M \in H$

$$M = \begin{pmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{pmatrix}$$

Let ${}^0H \subset H$ subspace of $N \in H$ s.t.

$$N = \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{pmatrix} \tag{1}$$

Fix $p \in E$. Then for $N \in {}^0H$

$$N_p = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \in M_{6\times 5}(k)$$

Imposing that $\operatorname{rank} N_p < 5$ gives two conditions on N

$$2 \leq \operatorname{codim}_{{}^0H} \underbrace{\{N \in {}^0H \mid \operatorname{rank} N_p < 5\}}_{{}^0H_p}$$

Since A is elliptic: E is one dimensional:

$$\bigcup_{p \in E} {}^0H_p \subsetneq {}^0H \subset H$$

□

Let $I \in \mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$. What is $\dim_k \mathsf{Ext}^1_A(I,I)$?

- For $M, N \in \mathsf{grmod}\, A$:

  $\sum_i (-1)^i h_{\underline{\mathsf{Ext}}^i_A(M,N)}(t) = h_M(t^{-1}) h_N(t) h_A(t^{-1})^{-1}$

- In particular: $\mathsf{pd}\, I = 1$, $\mathsf{Hom}_A(I,I) = k$,

$$h_I(t) = h_A(t) - \frac{s_I(t)}{1-t}$$

- Combining: (for $n > 0$)

$$\dim_k \mathsf{Ext}^1_k(I,I) = 1 + n + c$$

  where c is the constant term of

$$(t^{-1} - t^{-2}) s_I(t^{-1}) s_I(t)$$

- $\dim_k \mathsf{Ext}^1_k(I,I) \le 2n$ and equal iff $s_I(t)$ is