Ideals of three dimensional Artin-Schelter regular algebras

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February 17, 2006

Polynomial ring

Put $k = \mathbb{C}$. Commutative polynomial ring

$$S = k[x, y, z] = k\langle x, y, z\rangle/(f_1, f_2, f_3)$$

$$\begin{cases} f_1 : xy - yx = 0 \\ f_2 : yz - zy = 0 \\ f_3 : zx - xz = 0 \end{cases}$$

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Noncommutative polynomial rings

How to define them?
Pick certain properties of S.

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Noncommutative polynomial rings

Artin-Schelter (1986) defined class of algebras.

- A is quadratic: $A = k\langle x,y,z\rangle/(g_1,g_2,g_3)$

  Generic: $\begin{cases} g_1 : ayz + bzy + cx^2 = 0 \\ g_2 : azx + bxz + cy^2 = 0 \\ g_3 : axy + byx + cz^2 = 0 \end{cases}$

- A is cubic: $A = k\langle x,y\rangle/(g_1,g_2)$

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In what follows A will be (generic) quadratic. (some similar results for cubic)

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Homogeneous coordinate ring $S = k[x, y, z]$

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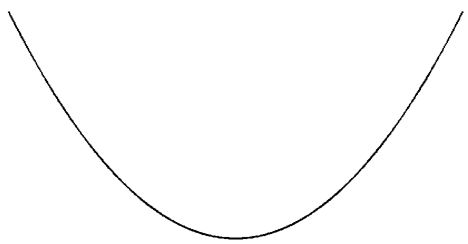
What has $S = k[x, y, z]$ to do with $\mathbb{P}^2$?

For any homogeneous polynomial $f \in k[x, y, z]$

$$\{p \in \mathbb{P}^2 \mid f(p) = 0\}$$

is a curve on $\mathbb{P}^2$.

Example:



$y^2 - xz$

Projective plane

Consider the projective plane $\mathbb{P}^2$.
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$S_d = \{$homogeneous polynomials degree $d\}$
$S = k \oplus S_1 \oplus S_2 \oplus \dots$ graded k-algebra

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$\mathbb{P}^2$ is completely determined by S.
Theorem of Serre (1955)

$$\operatorname{Qcoh} \mathbb{P}^2 \simeq \operatorname{GrMod} S / \operatorname{Tors} S$$

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What is GrMod S?

An object of GrMod S is

$$M = \dots \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus \dots$$

where

- M_d is k-vector space
- action of S on M such that $M_i S_j \subset M_{i+j}$

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What is Tors S?

Generated by modules $M \in \operatorname{GrMod} S$ for which

$$\forall m \in M : mS_{\geq d} = 0 \text{ for some } d$$

Typical:

$$M = \cdots \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus \cdots \oplus 0 \oplus 0 \oplus \dots$$

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Noncommutative projective plane

Model of noncommutative projective plane $\mathbb{P}^2_{\mathsf{q}}$
Artin-Zhang (1994)

- Replace S by noncommutative k-algebra A
- Define $\mathsf{Qcoh}\,\mathbb{P}^2_{\mathsf{q}} := \mathsf{GrMod}\,A/\,\mathsf{Tors}\,A$

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- Define $\text{Qcoh}\,\mathbb{P}^2_{\mathrm{q}} := \text{GrMod}\,A/\,\text{Tors}\,A$
- Arguments for taking A a quadratic Artin-Schelter algebra.

The points on $\mathbb{P}^2$

Point $p \in \mathbb{P}^2$

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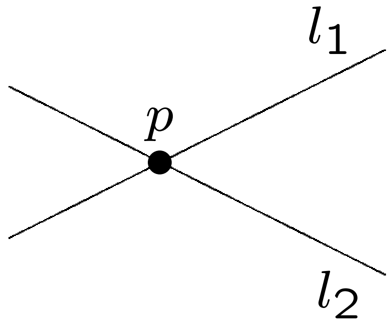
Point $p \in \mathbb{P}^2 \leftrightarrow$ two linear forms $l_1, l_2 \in S_1$



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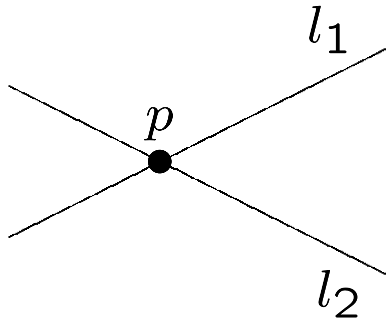

represented by

$$0 \to S(-2) \xrightarrow{\begin{pmatrix} -l_2 \\ l_1 \end{pmatrix} \cdot} S(-1)^2 \xrightarrow{\begin{pmatrix} l_1 & l_2 \end{pmatrix} \cdot} S \to P \to 0$$

where

- $P = P_0 \oplus P_1 \oplus P_2 \oplus \ldots \in \mathsf{GrMod}\, S$
- $P = P_0 S$
- $h_P(t) := \sum_d \dim_k P_d t^d = 1 + t + t^2 + \ldots$

P is called a point module.

The points on $\mathbb{P}^2$

Correspondence is reversible

$$\text{point } p \text{ on } \mathbb{P}^2 \;\leftrightarrow\; S\text{-module } P = P_0 S, h_P(t) = \frac{1}{1-t}$$

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point p on $\mathbb{P}^2$ $\leftrightarrow$ S-module $P = P_0 S, h_P(t) = \dfrac{1}{1-t}$

The points on $\mathbb{P}^2_q$: by definition

"point" p on $\mathbb{P}^2_q :=$ right A-module $P = P_0 A, h_P(t) = \dfrac{1}{1-t}$

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$$\text{“point” } p \text{ on } \mathbb{P}^2_q := \text{right } A\text{-module } P = P_0 A, h_P(t) = \frac{1}{1-t}$$

Artin, Tate and Van den Bergh (1990):

- There is divisor $E \subset \mathbb{P}^2$ of deg 3 such that

  (closed) point p on $E \leftrightarrow$ “point” on $\mathbb{P}^2_q$

- A, $\mathbb{P}^2_q$ determined by the “points” on $\mathbb{P}^2_q$

Generic: E is smooth elliptic curve

$$(a^3 + b^3 + c^3)xyz = abc(x^3 + y^3 + z^3)$$

The points on $\mathbb{P}^2$ versus graded S-ideals

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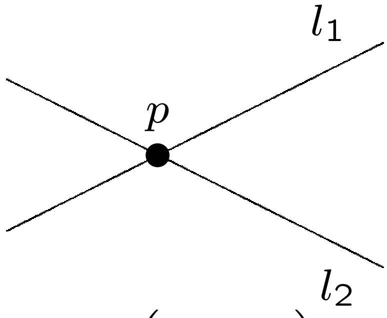

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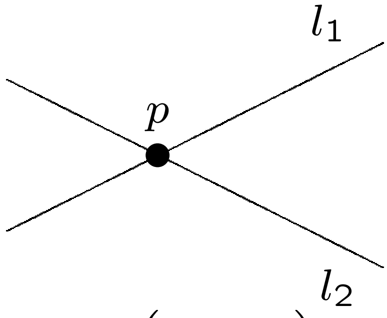

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$$I$$

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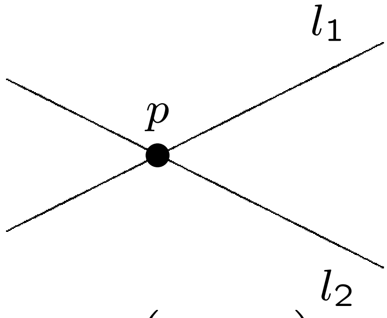

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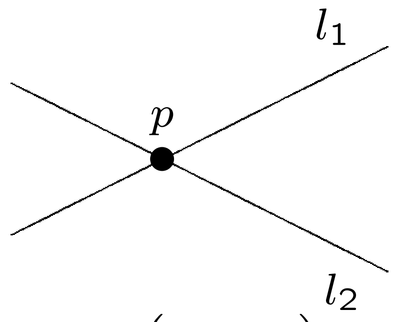

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If $I \subset S$ graded S-ideal:
- put $J = \omega\pi I$
- Either $J \cong S(d)$ or $\operatorname{pd} J = 1$.
- If $\operatorname{pd} J = 1$ then $J^{**} \cong S(d)$ i.e. $\underline{\operatorname{Ext}}^1_S(P, J) \neq 0$ for some point module P.

S-ideals of projective dimension one

Let $I \subset S$ graded ideal, $\operatorname{pd} I = 1$

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Known:
Given a_i, b_i there is such an ideal I (up to shift) if and only if

$$\deg\left(\oplus S(-i)^{b_i} \to \oplus_i S(-i)^{a_i}\right) = \begin{pmatrix} \times & \times & \dots & \times & \times \\ \times & \times & & \times & \times \\ & \times & & \times & \times \\ & & \ddots & \times & \times \\ & & & \times & \times \\ & & & & \times \end{pmatrix} \quad \times' s > 0$$

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if and only if

$$h_I(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

for some Castelnuovo polynomial $s(t)$.

A Castelnuovo polynomial is of the form

$$s(t) = 1 + 2t + 3t^2 + \cdots + ut^{u-1} + s_u t^u + \cdots + s_v t^v$$
$$u \quad \geq \quad s_u \geq \ \ldots \ \geq s_v \geq 0$$

for some integers $u, v \geq 0$.

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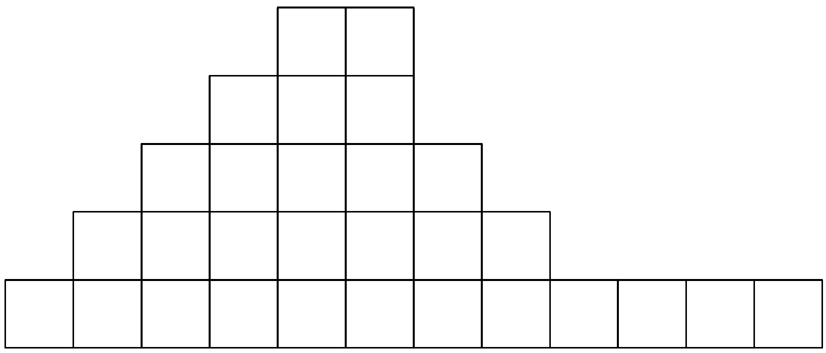
Visualized in form of a stair

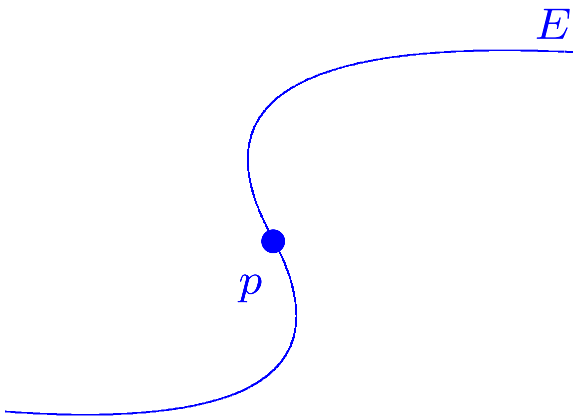
Example:



1 2 3 4 5 5 3 2 1 1 1 1

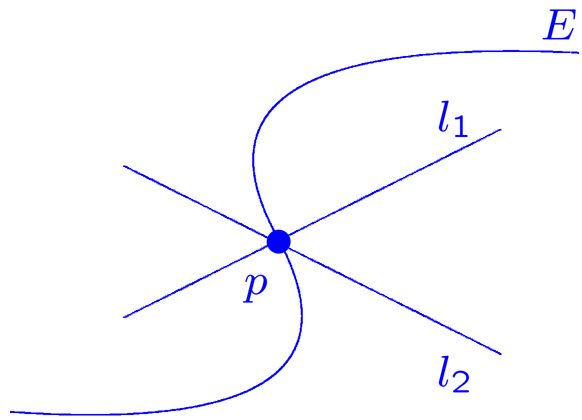
The "points" on $\mathbb{P}^2_q$ versus right A-ideals

"point" on $\mathbb{P}^2_q$



The "points" on $\mathbb{P}^2_q$ versus right A-ideals

"point" on $\mathbb{P}^2_q$ $\leftrightarrow$ two linear forms $l_1, l_2 \in A_1$ intersecting at E



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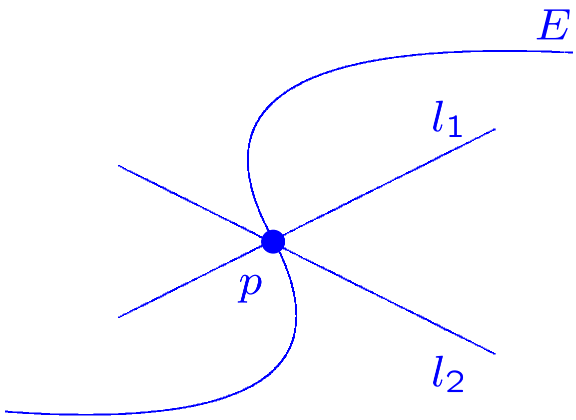

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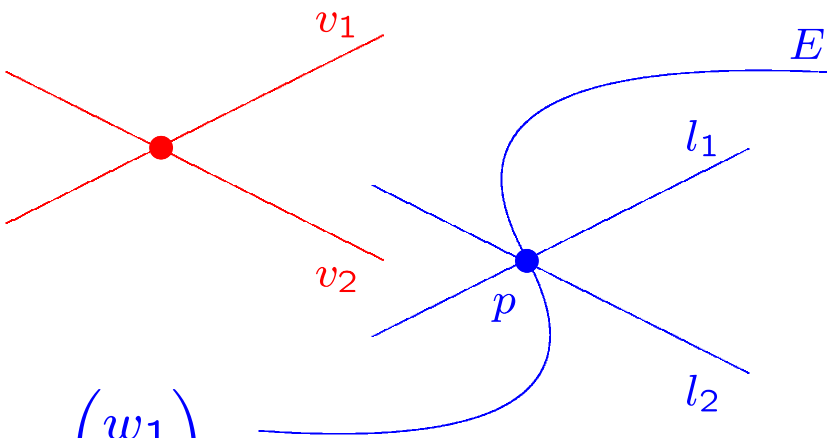

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$$\begin{array}{ccc} A(-1)^2 & & A \\ & \searrow \quad \nearrow & \\ & I & \end{array}$$

If $v_1, v_2 \in A_1$ <u>not</u> intersecting at E

$$0 \to A(-2) \xrightarrow{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot} A(-1)^2 \to I' \to 0$$

Then $\underline{\mathrm{Ext}}^1_A(P, I') = 0$ for all point modules P.
Such ideals I' are called <u>reflexive</u>.

Right A-ideals of projective dimension one

Let $I \subset A$ graded right ideal, $\operatorname{pd} I = 1$

$$0 \to \oplus_i A(-i)^{b_i} \xrightarrow{M\cdot} \oplus_i A(-i)^{a_i} \to I \to 0$$

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Given a_i, b_i there is such I (up to shift) if and only if (Theorem 6)

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If A generic: also true for reflexive ideals

Hilbert scheme of points on $\mathbb{P}^2$

Classify all possible configurations of n points on $\mathbb{P}^2$. Can be done by parameterspace.

Formally: parameter space for subschemes of $\mathbb{P}^2$ of dimension zero and degree n.

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moduli problem

$$\begin{aligned}\mathcal{H}ilb_n(\mathbb{P}^2) : \text{Noeth}\,/k &\to \text{Sets}\\ R &\mapsto \mathcal{H}ilb_n(\mathbb{P}^2)(R)\end{aligned}$$

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The functor $\mathcal{H}ilb_n(\mathbb{P}^2)$ is representable by projective variety $\text{Hilb}_n(\mathbb{P}^2) = \mathcal{H}ilb_n(\mathbb{P}^2)(k)$

- smooth
- connected
- dimension $2n$

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Initial problem: $\mathbb{P}^2_q$ has few zero-dimensional noncommutative subschemes

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$$\mathcal{H}ilb_n(\mathbb{P}^2_{\mathsf{q}})(R) = \{\mathcal{I} \in \mathrm{coh}\,\mathbb{P}^2_{\mathsf{q},R} \mid \mathcal{I} \text{ is } R\text{-flat and } \forall x \in \mathrm{Spec}\,R$$
$$\mathcal{I}_x \in \mathrm{coh}\,\mathbb{P}^2_{\mathsf{q},k(x)} \text{ torsion free, pd}\,1\text{, normalized, rk }1\}/\,\mathrm{Pic}\,R$$

In case $A = S$: agrees with $\mathcal{H}ilb_n(\mathbb{P}^2)$.

Hilbert scheme of points on $\mathbb{P}^2_q$

Initial problem: $\mathbb{P}^2_q$ has few zero-dimensional noncommutative subschemes
Solution: consider ideal sheaves instead
moduli problem

$$\mathcal{H}ilb_n(\mathbb{P}^2_q) : \text{Noeth}\,/k \to \text{Sets}$$
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In case $A = S$: agrees with $\mathcal{H}ilb_n(\mathbb{P}^2)$.

Nevins and Stafford (2002)
The functor $\mathcal{H}ilb_n(\mathbb{P}^2_q)$ is representable by projective variety $\text{Hilb}_n(\mathbb{P}^2_q) = \mathcal{H}ilb_n(\mathbb{P}^2_q)(k)$
- smooth
- dimension $2n$
- connectedness proved for almost all A
(using deformation theory and $\text{Hilb}_n(\mathbb{P}^2)$)

- A graded (right) ideal I is reflexive if

$$\underline{\mathrm{Ext}}^1(P, I) = 0 \text{ for all point modules } P$$

- If I is reflexive then $\mathrm{pd}\, I \leq 1$.

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reflexive graded S-ideals: $S(d)$

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---

reflexive graded S-ideals: $S(d)$

reflexive graded right A-ideals

In case A is generic: Theorems 1,2

$$\mathcal{R}(A) = \{\text{reflexive graded right } A\text{-ideals}\}/\text{iso,shift}$$

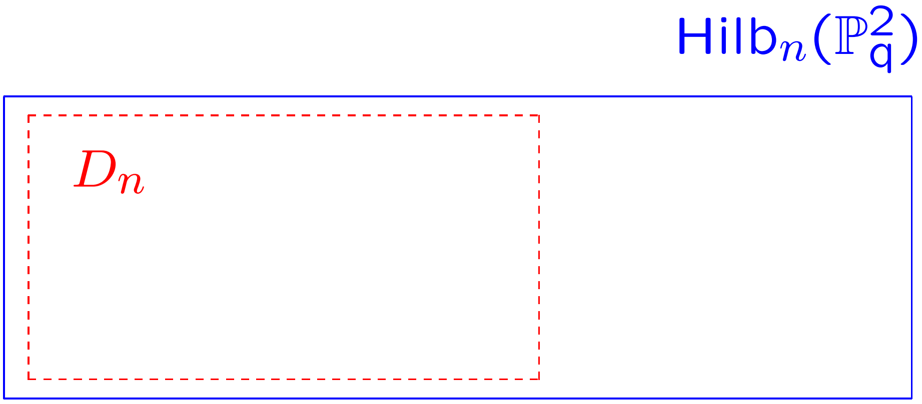
$$\updownarrow$$

$\coprod_n D_n$ where D_n is smooth affine variety of dim $2n$

points D_n are given by stable representations

$$k^n \overset{X}{\underset{Z}{\overset{\xrightarrow{\quad}}{\underset{\xrightarrow{\quad}}{\xrightarrow{Y}}}}} k^n \qquad \text{with rank} \begin{pmatrix} cX & aZ & bY \\ bZ & cY & aX \\ aY & bX & cZ \end{pmatrix} \le 2n+1$$

Picture in case A is generic:



- Hilb$_n(\mathbb{P}^2_{\mathsf{q}})$ parameterizes

  $\{$graded right A-ideals $I, \operatorname{pd} I = 1$

  $$h_I(t) \text{ is } \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \text{ up to shift}\}/\text{iso, shift}$$

- D_n parameterizes

  $\{$reflexive graded right A-ideals I

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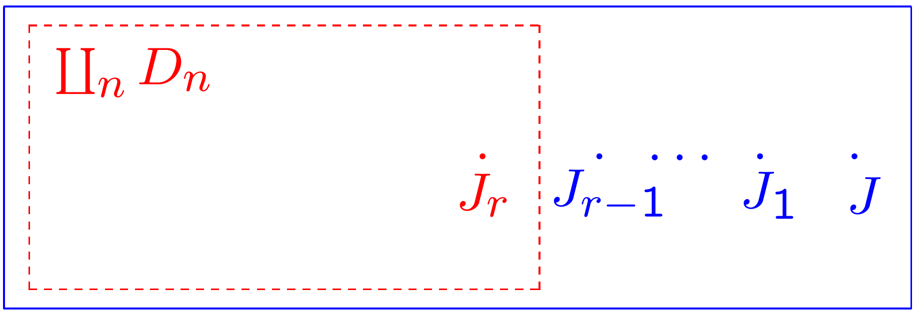
What about the boundary?

For any graded right ideal J with pd $J = 1$

$$\begin{aligned}
0 \to J \to &J_1 \to P_1(d_1) \to 0 \\
0 \to J_1 \to &J_2 \to P_2(d_2) \to 0 \\
&\vdots \\
0 \to J_{r-1} \to &J_r \to P_r(d_r) \to 0
\end{aligned}$$

where J_r is reflexive. Note $0 \leq r \leq n$.

$\coprod_n \mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$



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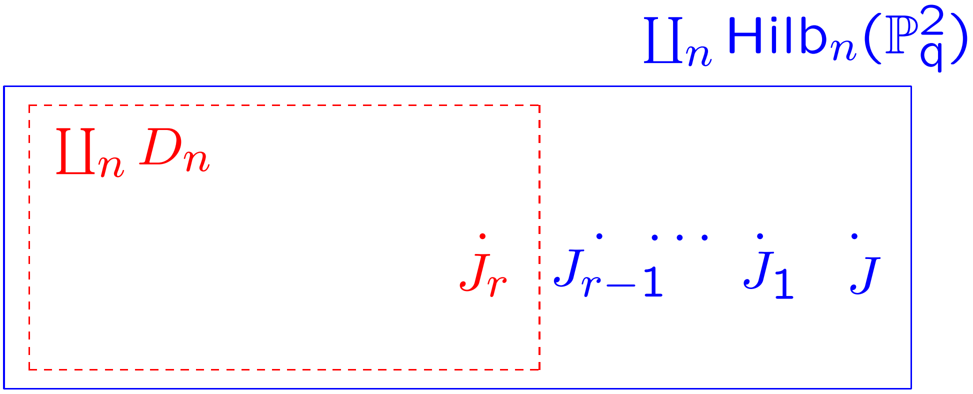

Let $\mathrm{Hilb}_n^{\ge d}(\mathbb{P}^2_q)$ be the $J \in \mathrm{Hilb}_n(\mathbb{P}^2_q)$ with $r \ge d$.

Theorem 7

- $\mathrm{Hilb}_n^{\ge d}(\mathbb{P}^2_q)$ projective variety of dimension $2n-d$
- boundary $\mathrm{Hilb}_n(\mathbb{P}^2_q) \setminus D_n$ has dimension $2n-1$

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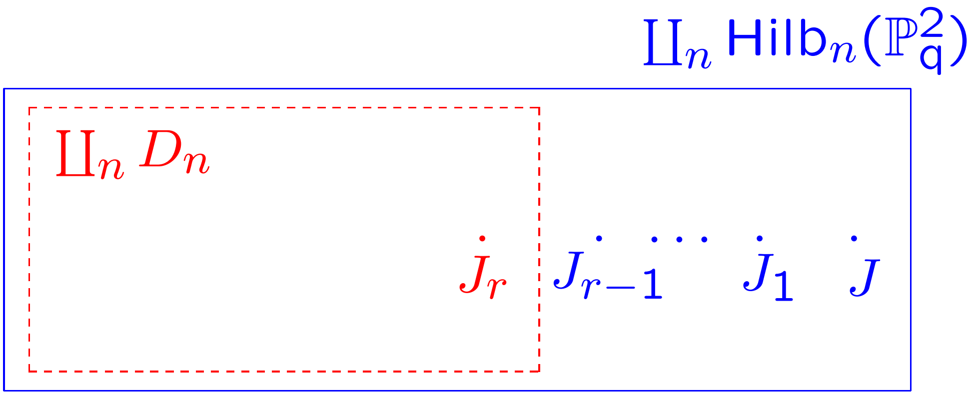

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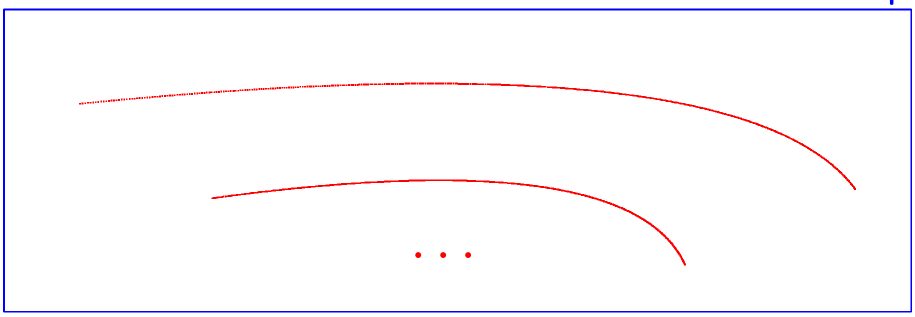
Thus the "actual" Hilbert scheme of "points" on $\mathbb{P}^2_q$ is $\mathrm{Hilb}_n^n(\mathbb{P}^2_q)$, has dimension n

Stratification of $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$

Consider all points of $\mathrm{Hilb}_n(\mathbb{P}^2)$ parameterizing ideals of A with same Hilbert series.

$Hilb_n(\mathbb{P}^2_{\mathsf{q}})$



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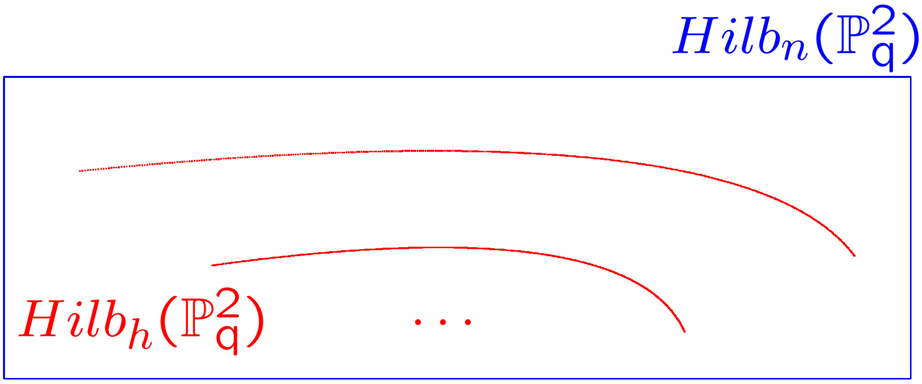
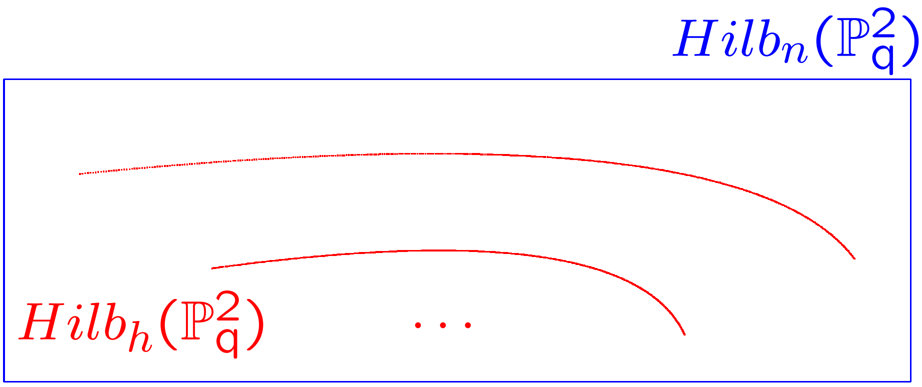


For an appearing Hilbert series

$$h(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

$s(t)$ is Castelnuovo polynomial, $s(1) = n$
put

$$\mathrm{Hilb}_h(\mathbb{P}^2_q) = \{I \in \mathrm{Hilb}_n(\mathbb{P}^2_q) \mid h_I(t) = h(t)\}$$

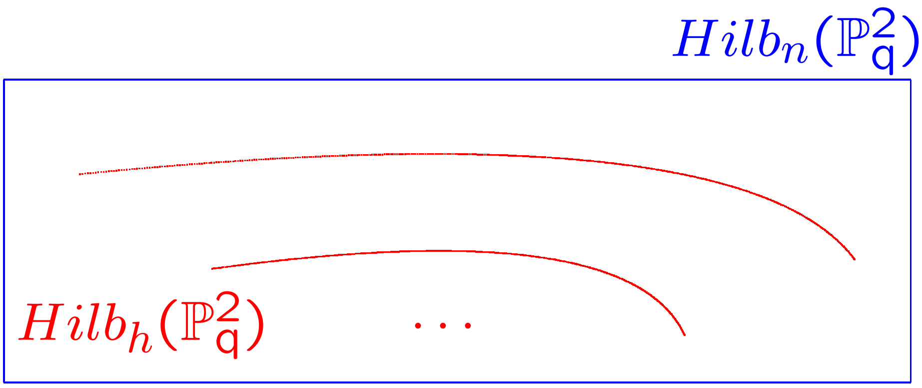


Chapter 3

$\text{Hilb}_h(\mathbb{P}^2_q) \subset \text{Hilb}_n(\mathbb{P}^2_q)$ is locally closed subvariety

- smooth
- connected

In case $A = S$: Proved by Gotzmann (1988)



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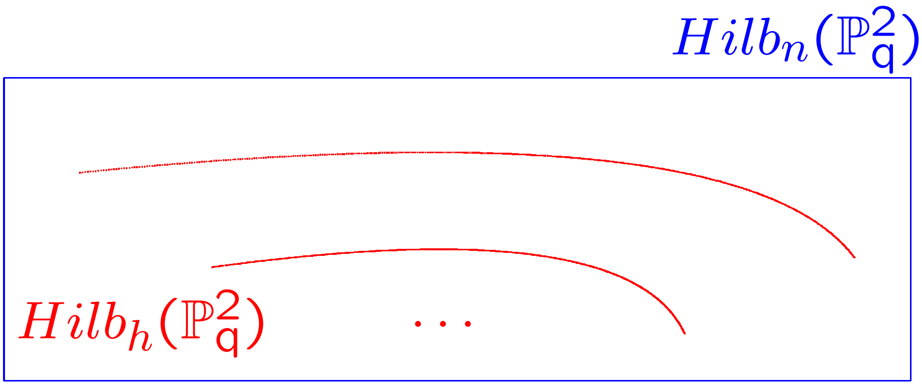
Formula for $\dim \mathrm{Hilb}_h(\mathbb{P}^2_{\mathsf{q}})$:

constant term of $(t^{-1} - t^{-2})s(t^{-1})s(t) + n + 1$

$\Rightarrow$ There is an unique stratum with maximal dimension $2n$



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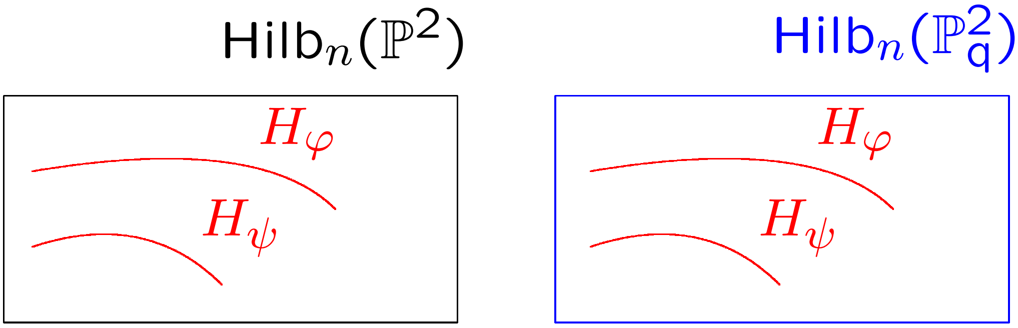
$\Rightarrow$ Theorem 5
$\mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$ is connected.

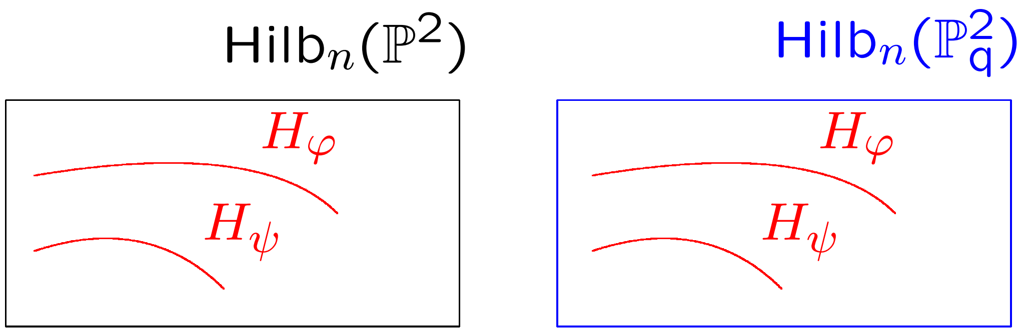
$\mathrm{Hilb}_n(\mathbb{P}^2)$ and $\mathrm{Hilb}_n(\mathbb{P}^2_q)$ analogous strata



- Incidence problem:

  for which φ, ψ do we have $H_\varphi \subset \overline{H_\psi}$?

$\mathrm{Hilb}_n(\mathbb{P}^2)$ and $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$ analogous strata



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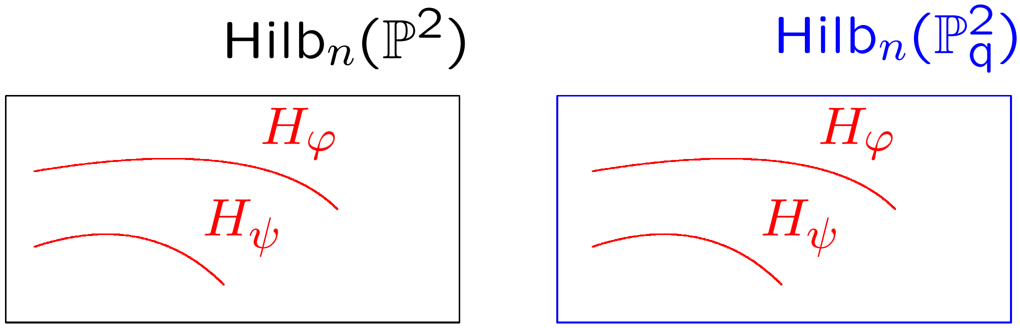
General incidence problem for $\mathrm{Hilb}_n(\mathbb{P}^2)$: unknown

Hilb$_n(\mathbb{P}^2)$ and Hilb$_n(\mathbb{P}^2_{\mathsf{q}})$ analogous strata



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General incidence problem for Hilb$_n(\mathbb{P}^2)$: unknown

- Guerimand (2002)
  solved case φ and ψ are "as close as possible"
  under a technical condition

“as close as possible” means: writing

$$\varphi(t) = \frac{1}{(1-t)^3} - \frac{s_\varphi}{1-t}, \quad \psi(t) = \frac{1}{(1-t)^3} - \frac{s_\psi}{1-t}$$

s_ψ is obtained from s_φ by minimal movement of one square to the left.

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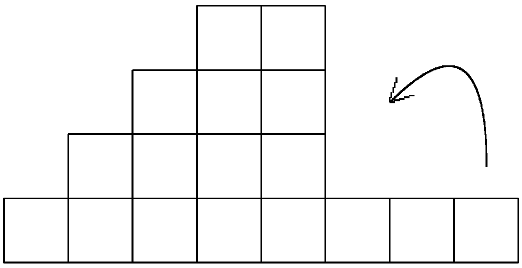
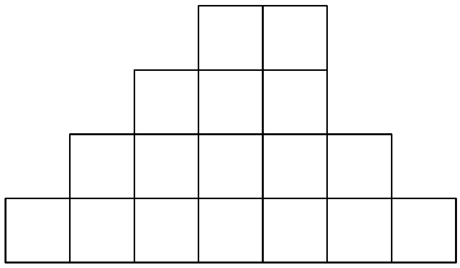
Examples:

$n = 17$: φ and ψ as close as possible



s_φ



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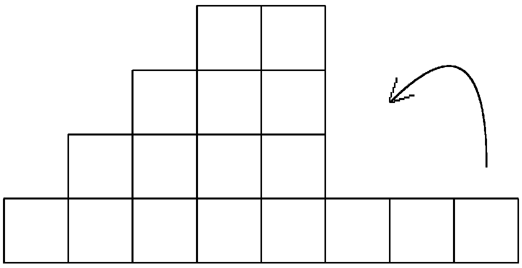
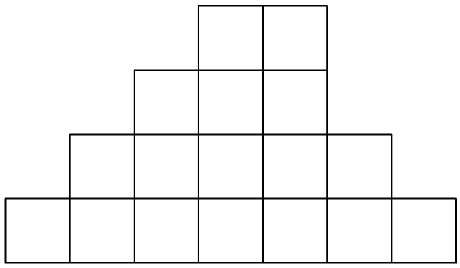
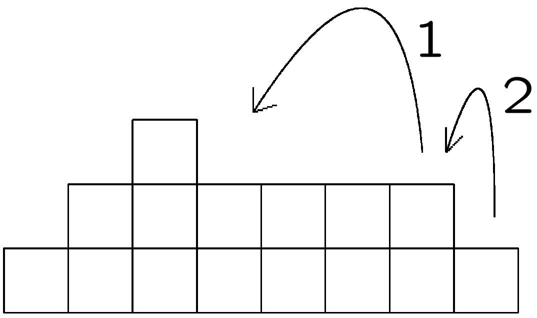


s_φ

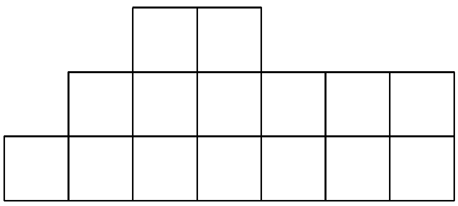


s_ψ

$n = 15$: φ and ψ not as close as possible



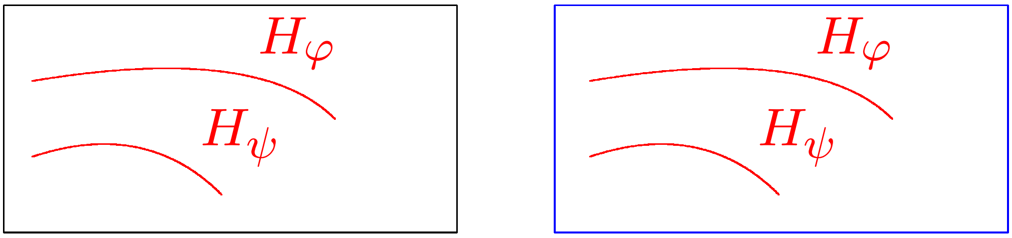
s_φ



s_ψ

$\mathrm{Hilb}_n(\mathbb{P}^2)$ and $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$ analogous strata

$\mathrm{Hilb}_n(\mathbb{P}^2)$ $\mathrm{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$



- Incidence problem:

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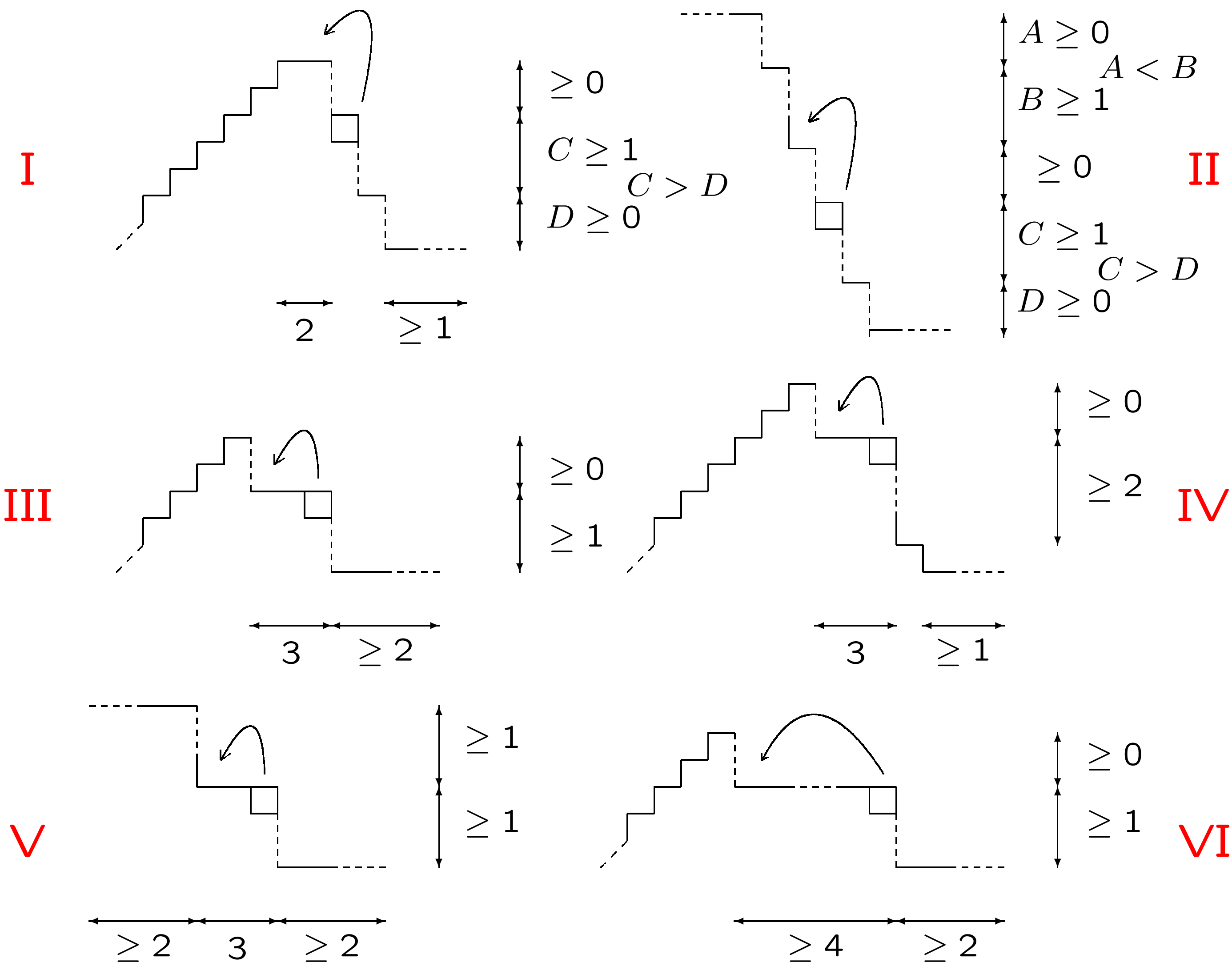
General incidence problem for $\mathrm{Hilb}_n(\mathbb{P}^2)$: unknown

- Guerimand (2002)
  solved case φ and ψ are "as close as possible"
  under a technical condition

- Theorem 9
  solved case φ and ψ are "as close as possible"

If φ, ψ Hilbert series "as close as possible":

We have $H_\varphi \subset \overline{H_\psi}$ if and only if



Before: s_φ, after: s_ψ

- Incidence problem:

  for which φ, ψ do we have $H_\varphi \subset \overline{H_\psi}$?

- Same solution for $\mathsf{Hilb}_n(\mathbb{P}^2_{\mathsf{q}})$ as for $\mathsf{Hilb}_n(\mathbb{P}^2)$?

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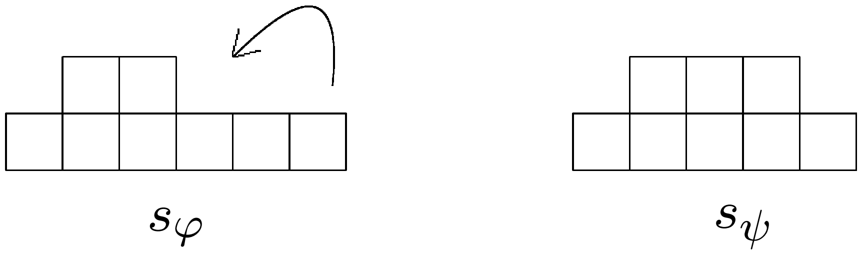
No for generic A



Generic $I \in H_\varphi$

$$0 \to A(-4) \oplus A(-7) \to A(-2) \oplus A(-3) \oplus A(-6) \to I \to 0$$

$$\mathcal{V} = \{ f : I \twoheadrightarrow F \mid h_F = \varphi - \psi \}$$

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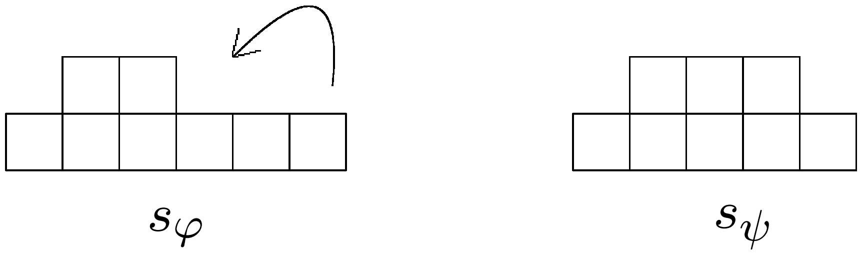

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- commutative case: $\mathcal{V} = \mathbb{P}^2$

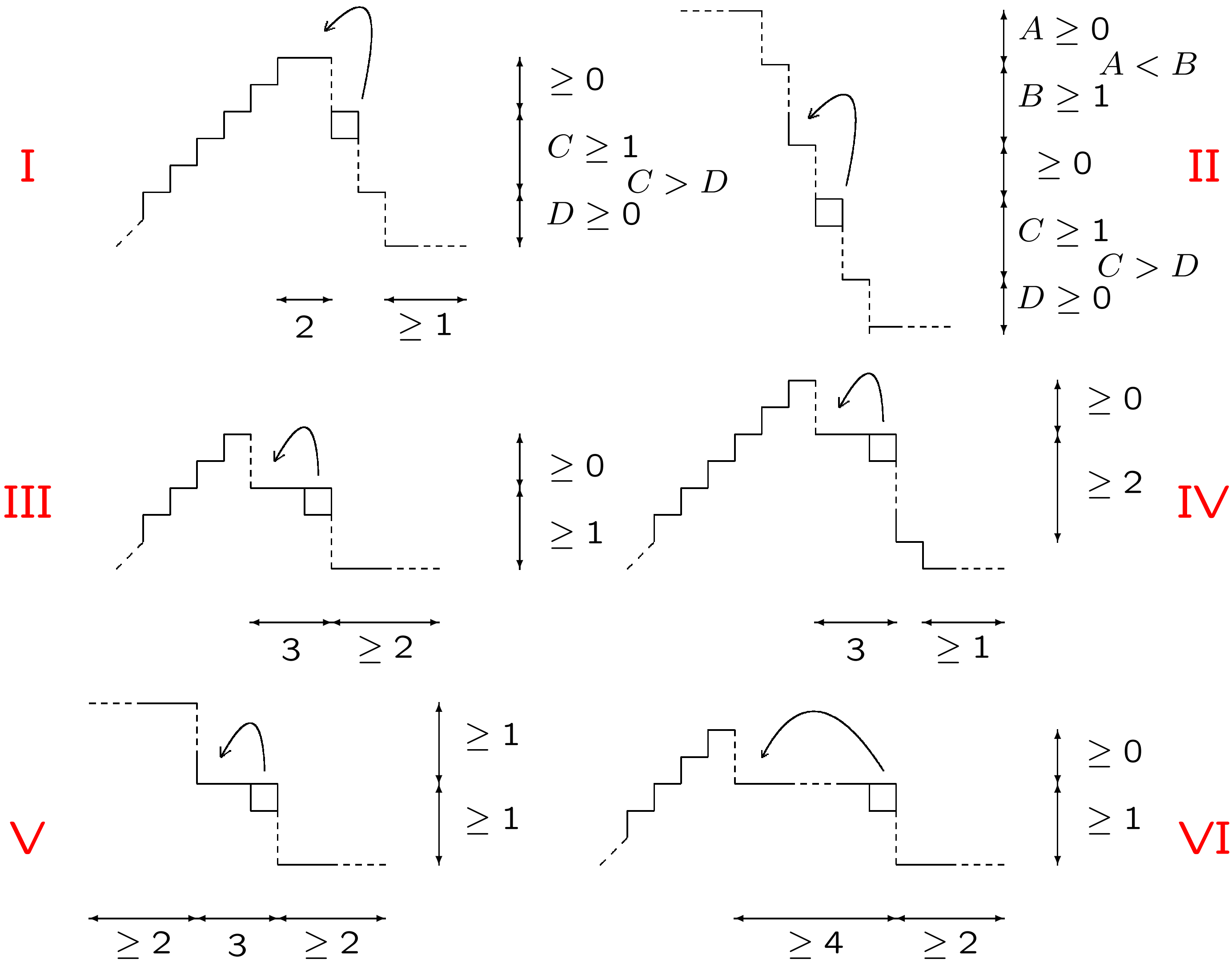
$\mathcal{V} \to \mathbb{N} : f \mapsto \dim_k \text{Ext}^1_A(\ker f, \ker f)$ non-constant hence $H_\varphi \subset \overline{H_\psi}$

- smooth elliptic case: $\mathcal{V} =$ three points on E

$\mathcal{V} \to \mathbb{N} : f \mapsto \dim_k \text{Ext}^1_A(\ker f, \ker f)$ constant hence $H_\varphi \not\subset \overline{H_\psi}$

If φ, ψ Hilbert series “as close as possible”:

We have $H_\varphi \subset \overline{H_\psi}$ if and only if



$A = S$ commutative	I	II	III	IV	V	VI
A generic	I	II	III			VI