

HILBERT SERIES OF MODULES OF GK-DIMENSION TWO OVER ELLIPTIC ALGEBRAS

KOEN DE NAEGHEL

ABSTRACT. We characterize the Hilbert functions and minimal resolutions of (critical) Cohen-Macaulay graded right modules of Gelfand-Kirillov dimension two over generic quadratic and cubic three dimensional Artin-Schelter regular algebras.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. In this paper we completely characterize the Hilbert series of critical graded Cohen-Macaulay modules of GK-dimension two over generic elliptic three dimensional Artin-Schelter regular algebras which are generated in degree one (see Theorem B below). Such modules can be viewed as irreducible curves in non-commutative deformations of \mathbb{P}^2 .

Our results complete a project started by Ajitabh [1, 2]. They form a natural counterpart to [12, 10] where one describes the possible Hilbert series for modules of GK-dimension three.

1.2. Elliptic algebras. In this paper k is the field of complex numbers \mathbb{C} .

We will be dealing with three dimensional Artin-Schelter regular k -algebras [4]. These graded connected algebras were classified in [5, 6, 16, 17] and have all expected nice homological properties. For example they are both left and right noetherian domains with global dimension three and Gelfand-Kirillov dimension three.

We will furthermore assume A to be generated in degree one, and we require that A is *generic* by which we mean that in the triple (E, \mathcal{L}, σ) associated to A [6],

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E is a smooth elliptic curve and σ is a translation under the group law of E of infinite order. This is equivalent [5] with saying that A takes one of the following forms:

- A is quadratic:

$$A = k\langle x, y, z \rangle / (f_1, f_2, f_3)$$

where f_1, f_2, f_3 are the homogeneous quadratic relations

$$\begin{cases} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{cases}$$

where $(a, b, c) \in \mathbb{P}^2$ for which $abc \neq 0$ and $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$.

- A is cubic:

$$k\langle x, y \rangle / (g_1, g_2)$$

where g_1, g_2 are the homogeneous cubic relations

$$\begin{cases} g_1 = ay^2x + byxy + axy^2 + cx^3 \\ g_2 = ax^2y + bxyx + ayx^2 + cy^3 \end{cases}$$

where $(a, b, c) \in \mathbb{P}^2$ for which $abc \neq 0$, $b^2 \neq c^2$ and $(2bc)^2 \neq (4a^2 - b^2 - c^2)^2$.

In this case A contains a central element g (which is unique up to scalar multiplication), homogeneous of degree three if A is quadratic and of degree four if A is cubic [5]. For later use we put r_A equal to the number of generators of A i.e.

$$r_A = \begin{cases} 3 & \text{if } A \text{ is quadratic} \\ 2 & \text{if } A \text{ is cubic} \end{cases}$$

For the rest of this paper, we will assume A to be such a generic three dimensional Artin-Schelter regular algebra, either quadratic or cubic. In the quadratic case these algebras are so-called three-dimensional Sklyanin algebras (for which the translation σ has infinite order).

1.3. Some terminology. By an A -module we will mean a finitely generated graded right A -module. We write $\text{grmod}(A)$ for the category of A -modules. For an A -module M and $n \in \mathbb{Z}$, write $M_{\leq n} = \bigoplus_{d \leq n} M_d$. Define $M(n)$ as the A -module equal to M with its original A action, but which is graded by $M(n)_i = M_{n+i}$. We refer to the modules $M(n)$ as *shifts (of grading)* of M . We say M is *normalized* if $M_{<0} = 0$ and $M_0 \neq 0$. The Hilbert series of M is denoted by

$$h_M(t) = \sum_{i=-\infty}^{+\infty} (\dim_k M_i) t^i \in \mathbb{Z}((t))$$

which makes sense since A is right noetherian and M is bounded below and locally finite. The Hilbert series of A is [4]

$$h_A(t) = \begin{cases} \frac{1}{(1-t)^3} & \text{if } A \text{ is quadratic} \\ \frac{1}{(1-t)^2(1-t^2)} & \text{if } A \text{ is cubic} \end{cases}$$

Taking Hilbert series of a projective (hence free) resolution it is easy to see that there exist integers r, a and a Laurent polynomial $s(t) \in \mathbb{Z}[t, t^{-1}]$ such that the Hilbert series of M is of the form

$$(1.1) \quad h_M(t) = h_A(t)(r + a(1-t) - s(t)(1-t)^2)$$

We write $\text{GKdim } M$ for the Gelfand-Kirillov dimension (GK-dimension for short) of M . As $\text{GKdim } A = 3$, $\text{GKdim } M \leq 3$ and it may be computed as the order of the pole of $h_M(t)$ at $t = 1$, see [6]. The leading coefficient e_M of the series expansion of $h_M(t)$ in powers of $1 - t$ is called the *multiplicity* of M . It is positive and by (1.1) an integer multiple of the multiplicity e_A of A , thus as in [6] it will be convenient to put $\iota_A = e_A^{-1}$ and $\epsilon_M = \iota_A e_M$ i.e.

$$\iota_A = 4 - r_A = \begin{cases} 1 & \text{if } A \text{ is quadratic} \\ 2 & \text{if } A \text{ is cubic} \end{cases} \quad \text{and } \epsilon_M = \begin{cases} e_M & \text{if } A \text{ is quadratic} \\ 2e_M & \text{if } A \text{ is cubic} \end{cases}$$

An A -module M is called *pure* if for all non-trivial submodules $N \subset M$ we have $\text{GKdim } N = \text{GKdim } M$. If in addition $e_N = e_M$ for all non-trivial submodules we say that M is *critical*. This is equivalent with saying that every proper quotient of M has lower GK-dimension. Any pure module M of GK-dimension d admits a filtration such that the successive quotients are critical of GK-dimension d . The graded Hom and Ext groups in $\text{grmod}(A)$ will be written as $\underline{\text{Hom}}$ and $\underline{\text{Ext}}$. We say that M is *Cohen-Macaulay* if $\text{pd } M = 3 - \text{GKdim } M$, or equivalently if $\underline{\text{Ext}}_A^i(M, A) = 0$ for $i \neq 3 - \text{GKdim } M$.

1.4. Modules of projective dimension one. In this paper we will be concerned with A -modules of projective dimension one. Such a module M admits a minimal resolution of the form

$$(1.2) \quad 0 \rightarrow \bigoplus_i A(-i)^{b_i} \rightarrow \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

The finitely supported sequences of non-negative integers $(a_i), (b_i)$ are usually called the (*graded*) *Betti numbers* of M . Taking Hilbert series of (1.2) one sees they are related to the Hilbert series of M by the formula

$$(1.3) \quad h_M(t) = h_A(t) \sum_i (a_i - b_i) t^i$$

The polynomial $q_M(t) = \sum_i (a_i - b_i) t^i \in \mathbb{Z}[t]$ is the so-called *characteristic polynomial* of M . We also write $p_M(t) = q_M(t)/(1 - t) \in \mathbb{Z}((t))$. Note that we have $p_M(t) \in \mathbb{Z}[t, t^{-1}]$ if and only if $\text{GKdim } M < 3$.

1.5. Main results. For an A -module M of GK-dimension two the following assertions are equivalent [6, §4]

- (1) M is pure of projective dimension one,
- (2) M has projective dimension one,
- (3) M is Cohen-Macaulay,
- (4) $M = M^{\vee\vee}$, where $M^\vee = \underline{\text{Ext}}_A^1(M, A)$.

Hence any A -module M of GK-dimension two is (uniquely) represented by a pure module of GK-dimension two and projective dimension one, namely $M^{\vee\vee}$.

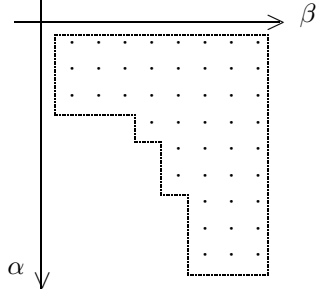
In order to state our main results, we will first need some terminology [12, §4.1]. For positive integers m, n consider the rectangle

$$R_{m,n} = [1, m] \times [1, n] = \{(\alpha, \beta) \mid 1 \leq \alpha \leq m, 1 \leq \beta \leq n\} \subset \mathbb{Z}^2$$

A subset $L \subset R_{m,n}$ is called a *ladder* if

$$\forall (\alpha, \beta) \in R_{m,n} : (\alpha, \beta) \notin L \Rightarrow (\alpha + 1, \beta), (\alpha, \beta - 1) \notin L$$

Example 1.1. The ladder below is indicated with a dotted line.



Let (c_i) be a finitely supported sequence of non-negative integers. We associate a sequence $S(c)$ of length $\sum_i c_i$ to (c_i) as follows

$$\dots, \underbrace{i-1, \dots, i-1}_{c_{i-1} \text{ times}}, \underbrace{i, \dots, i}_{c_i \text{ times}}, \underbrace{i+1, \dots, i+1}_{c_{i+1} \text{ times}}, \dots$$

where by convention the left most non-zero entry of $S(c)$ is the first term of the sequence $S(c)$.

To finitely supported sequences of integers $(a_i), (b_i)$ we associate the matrix $S = S(a, b) = (S(b)_\beta - S(a)_\alpha)_{\alpha, \beta}$. It has the properties of a “degree matrix”:

$$(1.4) \quad S_{\alpha+1, \beta} \leq S_{\alpha, \beta} \leq S_{\alpha, \beta+1} \quad \text{and} \quad S_{\alpha, \beta} - S_{\alpha, \beta'} = S_{\alpha', \beta} - S_{\alpha', \beta'}$$

from which it follows that

$$(1.5) \quad L_{a, b} = \{(\alpha, \beta) \in R_{m, n} \mid S(a)_\alpha < S(b)_\beta\} \text{ where } m = \sum_i a_i, n = \sum_i b_i$$

is a ladder. Our following main result is proved in §3 below.

Theorem A. *Let $(a_i), (b_i)$ be finitely supported sequences of integers and put $m = \sum_i a_i, n = \sum_i b_i$.*

- (1) $(a_i), (b_i)$ appear as the Betti numbers of a graded right A -module M of GK-dimension two and projective dimension one if and only if
 - (a) The $(a_i), (b_i)$ are non-negative.
 - (b) $\sum_i b_i = \sum_i a_i$.
 - (c) $\forall (\alpha, \beta) \in R_{n, m} : \beta \geq \alpha \Rightarrow (\alpha, \beta) \in L_{a, b}$.
- (2) $(a_i), (b_i)$ appear as the Betti numbers of a critical graded right A -module M of GK-dimension two and projective dimension one if and only if
 - (a) The $(a_i), (b_i)$ are non-negative.
 - (b) $\sum_i b_i = \sum_i a_i$.
 - (c) $\forall (\alpha, \beta) \in R_{n, m} : \beta \geq \alpha - 1 \Rightarrow (\alpha, \beta) \in L_{a, b}$.
 - (d) If A is cubic it is not true that $(n \geq 2 \text{ and } \forall \alpha, \beta : S(b)_\beta - S(a)_\alpha = 1)$.

In both statements, the module M may be chosen to be g -torsionfree.

Remark 1.2. For quadratic A it was proved in [1] that the appearing conditions in Theorem A are necessary, and in [2] there were shown to be sufficient in the case where $\sum_i b_i = \sum_i a_i = 1$.

Remark 1.3. Theorem A is an analogue of the description of the Betti numbers of pure A -modules of GK-dimension three and projective dimension one, see [12, 10].

By Theorem A, a minimal resolution of a (resp. critical) A -module M of GK-dimension two and projective dimension one is of the form

$$0 \rightarrow \bigoplus_i A(-i)^{b_i} \rightarrow \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

for which a generic map $f : \bigoplus_i A(-i)^{b_i} \rightarrow \bigoplus_i A(-i)^{a_i}$ is represented by left matrix multiplication with a matrix of the form

$$f = \begin{pmatrix} * & * & * & \dots & * \\ & * & * & \dots & * \\ & & * & \dots & * \\ & & & \ddots & \vdots \\ & & & & * \end{pmatrix} \text{ resp. } f = \begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ & * & * & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & * & * \end{pmatrix}$$

where the indicated entries $*$ are nonzero homogeneous elements in A of degree ≥ 1 . In case A is cubic, there is an additional condition for critical M : Not all entries in f have degree 1, unless f is a 1×1 matrix (reflecting condition (2)(d) in Theorem A). In other words, in case A is cubic then the minimal resolution of a critical graded right A -module M of GK-dimension two and projective dimension one cannot be the form (up to shift of grading)

$$0 \rightarrow A(-1)^n \rightarrow A^n \rightarrow M \rightarrow 0, \quad n \geq 2$$

This might seem surprising for the reader. The reason is explained in Example 3.3 below.

As a consequence of Theorem A, we will deduce in §4

Theorem B. *Let $\epsilon > 0$ be an integer and put $e = \epsilon/\iota_A$. There is a bijective correspondence between Hilbert series $h(t)$ of normalized A -modules of GK-dimension two, projective dimension one and multiplicity e , and polynomials $s(t) = \sum_i s_i t^i \in \mathbb{Z}[t]$ which satisfy*

$$(1.6) \quad \epsilon > s_0 \geq s_1 \geq \dots \geq 0$$

The correspondence is given by $h(t) = h_A(t)(\epsilon(1-t) - s(t)(1-t)^2)$, explicitly

$$(1.7) \quad h(t) = \begin{cases} \frac{e}{(1-t)^2} - \frac{s(t)}{1-t} & \text{if } A \text{ is quadratic} \\ \frac{2e}{(1-t)(1-t^2)} - \frac{s(t)}{1-t^2} & \text{if } A \text{ is cubic} \end{cases}$$

Further, if we restrict to critical A -modules then the same statement holds where (1.6) is replaced by

$$(1.8) \quad \epsilon > s_0 > s_1 > \dots \geq 0 \text{ and if } A \text{ is cubic and } \epsilon > 1 \text{ then } s(t) \neq 0$$

It is clear that there are only finitely many polynomials $s(t) \in \mathbb{Z}[t]$ which satisfy (1.8). Hence Theorem B implies that there are only finitely many possibilities for the Hilbert series of a critical normalized Cohen-Macaulay A -module of GK-dimension two and multiplicity e . This consequence was already observed by Ajitabh in [1] for quadratic A . In fact it is easy to count the number of possibilities.

Corollary 1.4. *Let $\epsilon > 0$ be an integer and put $e = \epsilon/\iota_A$. The number of Laurent power series which appear as the Hilbert series of a critical normalized module of GK-dimension two, projective dimension one and multiplicity e is equal to*

$$\begin{cases} 2^{\epsilon-1} - 1 & \text{if } A \text{ is cubic and } \epsilon > 1, \\ 2^{\epsilon-1} & \text{else.} \end{cases}$$

Remark 1.5. It follows from Theorem B that there are infinitely many possibilities for the Hilbert series of a normalized Cohen-Macaulay module of GK-dimension two and multiplicity $e > 1$. This is to be expected, for example if A is quadratic and S is a line module over A then $M = S^{e-1} \oplus S(-n)$ is a (non-critical) normalized Cohen-Macaulay A -module GK-dimension two and multiplicity e , for all integers $n \geq 0$.

Remark 1.6. For the convenience of the reader we have included in Appendix A the list of possible Hilbert series and Betti numbers of critical normalized Cohen-Macaulay modules M of GK-dimension two and $\epsilon_M \leq 4$.

Remark 1.7. It is well-known that Theorem A (and hence Theorem B) holds for the commutative polynomial ring $k[x, y, z]$, which is a non-generic quadratic three dimensional Artin-Schelter regular algebra. See for example [9, Proposition 2.7 and Theorem 2.8]. We conjecture that Theorems A and B are true for all three dimensional Artin-Schelter regular algebras generated in degree one (thus not only the generic ones).

We end this introduction by saying a few words about the proof of Theorem A. The most difficult part is to show that the conditions in Theorem A(2) are sufficient §3.4. Roughly, this will be derived from the following three observations:

- To any g -torsionfree A -module M of GK-dimension two one may associate a divisor on the elliptic curve E , denoted by $\text{Div } M$. This notion was introduced by Ajitabh in [1], who showed in [2] that writing $\text{Div } M = D + (q)$ for some effective divisor D on E , there is a sufficient condition on D (called quantum-irreducibility) for M to be critical. See also §2.3 and §2.6 below.
- For any positive integer, there exists an effective multiplicity-free quantum-irreducible divisor on E . This was shown in [2]. See also §2.6 below.
- Let $(a_i), (b_i)$ be finitely supported sequences of integers satisfying Theorem A(2). Let D be a multiplicity-free effective divisor on E of degree $r_A \sum_i i(b_i - a_i) - 1$. In Theorem 3.5 below we show that there is a g -torsionfree $M \in \text{grmod}(A)$ of GK-dimension two which has a minimal resolution

$$0 \rightarrow \bigoplus_i A(-i)^{b_i} \xrightarrow{f} \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0,$$

for which the matrix representing the map f has the form

$$f = \begin{pmatrix} * & * & * & \dots & * & * \\ * & & & & & * \\ & * & & & & * \\ & & \ddots & & & \vdots \\ & & & * & & * \\ & & & & * & * \end{pmatrix}$$

(where the entries off the diagonal, first row and last column are zero) and for which $\text{Div } M = D + (q)$ for some $q \in E$.

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2. PRELIMINARIES

Throughout we will assume A to be a generic three-dimensional Artin-Schelter regular algebra, either quadratic or cubic, as described in §1.2.

2.1. Geometric data. In this part we recall some terminology and basic facts on elliptic algebras from [5, 6].

The algebra A is completely determined by geometric data (E, \mathcal{L}, σ) where

- if A is quadratic then $j : E \hookrightarrow \mathbb{P}^2$ is a divisor of degree three, $\mathcal{L} = j^* \mathcal{O}_{\mathbb{P}^2}(1)$ line bundle of degree three and $\sigma \in \text{Aut}(E)$,
- if A is cubic then $j : E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of bidegree $(2, 2)$, $\mathcal{L} = j^* \text{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ line bundle of degree two and $\sigma \in \text{Aut}(E)$.

As we choose A to be generic, E is smooth curve of arithmetic genus one i.e. an elliptic curve, and σ is a translation on E . In case A is cubic then σ is of the form $\sigma(q_1, q_2) = (q_2, f(q_1, q_2))$ for some map $f : E \rightarrow \mathbb{P}^1$.

Let $\mathcal{E} \in \text{Pic}(E)$ be a line bundle on E . We use the notation \mathcal{E}^σ for the pull-back $\sigma^* \mathcal{E}$. Thus $(\mathcal{E}^\sigma)_p = \mathcal{E}_{p\sigma}$ for $p \in E$. We regard $\text{Pic}(E)$ as a module over $\mathbb{Z}[\sigma, \sigma^{-1}]$, where the action of a Laurent polynomial $f(\sigma) = \sum_i a_i \sigma^i$ on \mathcal{E} is defined as

$$\mathcal{E}^{f(\sigma)} := \otimes_i (\mathcal{E}^{\sigma^i})^{\otimes a_i}$$

Recall that there is, up to scalar multiplication, a canonical central element $g \in A$, homogeneous of degree $\iota_A r_A$. The factor ring A/gA is isomorphic to the twisted homogeneous coordinate ring

$$B = \bigoplus_{n \geq 0} H^0(E, \mathcal{L}_n) \text{ where } \mathcal{L}_n = \mathcal{L} \otimes_E \mathcal{L}^\sigma \otimes_E \cdots \otimes_E \mathcal{L}^{\sigma^{n-1}} = \mathcal{L}^{(1-\sigma^n)/(1-\sigma)}$$

is a line bundle of degree $r_A n$. Note that in case A is cubic we have $\mathcal{L} = j^* \text{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(1)$. Multiplication in B is defined by $b_n b_m = b_n \otimes_E b_m^{\sigma^n}$ for $b_n \in B_n$, $b_m \in B_m$, where $b_m^{\sigma^n} = b_m \circ \sigma^n$. The algebra B has Gelfand-Kirillov dimension two, and it is a domain since E is reduced. The homogeneous elements of B will be identified with the corresponding sections of the appropriate line bundles on E . For any $m \in A_n$, we denote by \overline{m} its image in $B \cong A/gA$.

There is a (left exact) global section functor

$$\Gamma_* : \text{coh}(E) \rightarrow \text{grmod}(B) : \mathcal{F} \rightarrow \bigoplus_{n \geq 0} H^0(E, \mathcal{F} \otimes_E \mathcal{L}_n)$$

whose right adjoint is exact, denoted by $\widetilde{(-)}$. It was shown in [7] that they induce a category equivalence between $\text{coh}(E)$ and $\text{grmod}(B)/\text{grmod}(B)_0$. Here $\text{grmod}(B)_0$ stands for the Serre subcategory of the finite length modules in the category $\text{grmod}(B)$ of finitely generated graded right B -modules.

It will be convenient below to let the shift functors $-(n)$ on $\text{coh}(E)$ be the ones obtained from the equivalence and not the ones coming from the embedding j . Thus $\mathcal{O}_E(n) = \sigma_*^n \mathcal{L}_n$ and $\mathcal{F}(n) = \sigma_*^n \mathcal{F} \otimes_E \mathcal{O}_E(n)$ for $\mathcal{F} \in \text{coh}(E)$.

For $p \in E$ we write $P = (\Gamma_*(k(p)))_A \in \text{grmod}(A)$ where $k(p)$ is the skyscraper sheaf k sitting at p . Observe that $k(p)(n) = k(p^{\sigma^n})$. Such A -modules P are called *point modules* over A . It is easy to see that $h_P(t) = (1-t)^{-1}$.

2.2. Group law and divisors on E . Fixing a group law on E the automorphism σ is a translation by some point $\xi \in E$. Thus $p^\sigma = p + \xi$ for $p \in E$. We write o for the origin of the group law. Linear equivalence of divisors D, D' on E will be denoted by $D \sim D'$. We will frequently use

Proposition 2.1. ([13, IV Theorem 4.13B]) *Let D, D' be two divisors on E . Then $D \sim D' \Leftrightarrow \deg D = \deg D'$ and D, D' have the same sum in the group law of E*

For example, for three points $p, q, r \in E$ we have $p = q + r$ in the group law of E if and only if $(p) + (o) \sim (q) + (r)$ as divisors on E .

For a nonzero global section $s \in H^0(E, \mathcal{E})$ and $p \in E$ we write $s(p)$ for the image of s in the one dimensional k -linear space $\mathcal{E} \otimes_E k(p) \cong \mathcal{E}_p/m_p \mathcal{E}_p$, where m_p is the maximal ideal of the local ring \mathcal{O}_p and $k(p) = \mathcal{O}_p/m_p$. In case $\mathcal{E} = \mathcal{O}_E(i)$ we have $s(p) \in \mathcal{O}_E(i) \otimes_E k(p) = k(p^{\sigma^i})$.

We write s^σ for the image of s under the k -linear isomorphism $H^0(E, \mathcal{E}) \cong H^0(E, \mathcal{E}^\sigma)$. We have $s^\sigma(p) = s(p^\sigma)$ under the isomorphism $k(p)^\sigma \cong k(p^\sigma)$. We write $\text{Div}(s)$ for the divisor of zeros of s . It follows that $\text{Div}(s^\sigma) = \sigma^{-1} \text{Div}(s)$.

Consider a map $N : \mathcal{O}_E(-j) \rightarrow \mathcal{O}_E(-i)$ where $i < j$. As $\mathcal{O}_E(-i) \otimes k(p^{\sigma^i}) = k(p)$, a point $p \in E$ is supported on the cokernel of $N = (n)$ if and only if $N \otimes k(p^{\sigma^i}) = (n^{\sigma^i}(p))$ is zero. Here, n is viewed as a global section of $\mathcal{O}_E(j-i)$.

This is generalized as follows. Let $(a_i), (b_i)$ be finitely supported sequences of non-negative integers. Consider a map $N : \bigoplus_i \mathcal{O}_E(-i)^{b_i} \rightarrow \bigoplus_i \mathcal{O}_E(-i)^{a_i}$. To $N = (n_{\alpha\beta})_{\alpha\beta}$ we associate a new matrix X_N , given by $(X_N)_{\alpha\beta} = n_{\alpha\beta}^{\sigma^{S(a)_\alpha}}$. It is easy to see that a point $p \in E$ is supported on the cokernel of $N = (n_{\alpha\beta})_{\alpha\beta}$ if and only if the rank of the matrix $X_N(p)$ is less than $\sum_i a_i$, where

$$X_N(p) := X_N \otimes k(p) \quad \text{i.e.} \quad (X_N(p))_{\alpha\beta} = n_{\alpha\beta}^{\sigma^{S(a)_\alpha}}(p)$$

2.3. The divisor of a curve module. By a *curve A -module* [1] we will mean a g -torsionfree A -module M of GK-dimension two.

It was shown in [1, 3] that to any curve A -module one may associate a divisor on E . Actually this was done in case A is quadratic, but a similar treatment holds for cubic A . Let us recall how this is done, considering both cases (quadratic and cubic) at the same time.

Let M be a curve A -module. As M is g -torsionfree, M/Mg has GK-dimension one. Hence $(M/Mg)^\sim$ is a finite dimensional \mathcal{O}_E -module which corresponds to a divisor on E . We will call this the *divisor* of M and denote it by $\text{Div}(M)$.

Proposition 2.2. ([3]) *Let M be curve A -module.*

- (1) $\text{Div}(M)$ is an effective divisor of degree $r_A \in M$.
- (2) For any integer l we have $\text{Div}(M(l)) = \sigma^l \text{Div}(M)$.
- (3) Div is additive on short exact sequences i.e. for a short exact sequence of curve modules $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ in $\text{grmod}(A)$ we have

$$\text{Div}(M) = \text{Div}(M') + \text{Div}(M'')$$

- (4) Let $p \in E$ and write $P = (\Gamma_*(\mathcal{O}_p))_A$ for the corresponding point module.

Assume we have an exact sequence $0 \rightarrow K \rightarrow M \xrightarrow{f} P$ where $f \neq 0$. Then K is a curve A -module and

$$\text{Div}(K) = \text{Div}(M) - (p) + (p^{\sigma^{-l}A^rA})$$

- (5) Let $p \in E$. If $\text{Hom}_A(M, P) \neq 0$ then $p \in \text{Supp}(\text{Div}(M))$. In case $M_{<0} = 0$ the converse is also holds.

We also mention

Lemma 2.3. *Let M be a pure curve A -module. Then $\text{Div}(M) = \text{Div}(M^{\vee\vee})$.*

Proof. By [6, Corollary 4.2] the canonical map $\mu_M : M \rightarrow M^{\vee\vee}$ is injective and its cokernel is finite dimensional. Thus $\pi M = \pi M^{\vee\vee}$ and hence $(M/Mg)^\sim = (M^{\vee\vee}/M^{\vee\vee}g)^\sim$. This means that $\text{Div}(M) = \text{Div}(M^{\vee\vee})$. \square

For any g -torsionfree $a \in A_n$ the divisor $\text{Div}(A/aA)$ of the curve A -module $M = A/aA$ coincides with the divisor of zeros $\text{Div}(\bar{a})$ of the global section $\bar{a} \in H^0(E, \mathcal{L}_n)$. Indeed, this follows from the short exact sequence in $\text{coh}(E)$

$$0 \rightarrow \mathcal{O}_E(-n) \xrightarrow{\bar{a}} \mathcal{O}_E \rightarrow (M/Mg)^\sim \rightarrow 0$$

More generally, in [1] it was shown that for any Cohen-Macaulay curve A -module M we may interpret $\text{Div}(M)$ as the divisor of zeros of some global section $s_{[M]}$ of the invertible sheaf $\mathcal{L}^{p_M(\sigma)}$ on E . As this will play a key role further on, we will now recall the construction of $s_{[M]}$. For more details the reader is referred to [1, 2].

Let M be a Cohen-Macaulay curve A -module, say with minimal resolution

$$(2.1) \quad 0 \rightarrow \bigoplus_i A(-i)^{b_i} \xrightarrow{f} \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

We represent the map f in (2.1) by left multiplication by a matrix $[M]$ whose entries are homogeneous elements $m_{\alpha\beta}$ in A . Applying the functor $- \otimes_A B$ to (2.1) we find an exact sequence in $\text{grmod}(B)$

$$(2.2) \quad 0 \rightarrow \bigoplus_i B(-i)^{b_i} \xrightarrow{\bar{f}} \bigoplus_i B(-i)^{a_i} \rightarrow M/Mg \rightarrow 0$$

where we have used the g -torsionfreeness of M to derive

$$\text{Tor}_1^A(M, A/Ag) = \ker(M(-\iota_{AR}A) \xrightarrow{g} M) = 0$$

The map \bar{f} is represented by $[\overline{M}]$, the matrix obtained from $[M]$ by replacing the entries $m_{\alpha\beta} \in A$ by $\overline{m}_{\alpha\beta} \in B$. Applying the exact functor $(-)$ on (2.2) we obtain an exact sequence in $\text{coh}(E)$

$$(2.3) \quad 0 \rightarrow \bigoplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{[\overline{M}]} \bigoplus_i \mathcal{O}_E(-i)^{a_i} \rightarrow (M/Mg)^\sim \rightarrow 0$$

It is now clear that the divisor of $(M/Mg)^\sim$ is precisely the zerodivisor of $\det(X_{[M]})$, where as in §2.2 the matrix $X_{[M]}$ is defined as

$$X_{[M]} = (\overline{m}_{\alpha\beta}^{\sigma^{S(a)}\alpha})_{\alpha\beta} \text{ where } \overline{m}_{\alpha\beta}^{\sigma^{S(a)}\alpha} \in H^0(E, \mathcal{L}_{S(b)\beta - S(a)\alpha}^{\sigma^{S(a)}\alpha})$$

and

$$\det(X_{[M]}) = \sum_{\gamma \in S_r} \text{sgn}(\gamma) \overline{m}_{1\gamma(1)}^{\sigma^{S(a)}_1} \otimes \overline{m}_{2\gamma(2)}^{\sigma^{S(a)}_2} \otimes \cdots \otimes \overline{m}_{r\gamma(r)}^{\sigma^{S(a)}_r}$$

We denote $s_{[M]} = \det X_{[M]}$.

Example 2.4. Consider a Cohen-Macaulay curve A -module with minimal resolution of the form

$$0 \rightarrow A(-3) \oplus A(-7) \oplus A(-8) \xrightarrow{[M]} A(-1) \oplus A(-2) \oplus A(-7) \rightarrow M \rightarrow 0$$

where

$$[M] = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{pmatrix}$$

where the entries $m_{ij} \in A$ are homogeneous elements with appropriate degrees. The corresponding exact sequence on $\text{coh}(E)$ becomes

$$0 \rightarrow \mathcal{O}_E(-3) \oplus \mathcal{O}_E(-7) \oplus \mathcal{O}_E(-8) \xrightarrow{[M]} \mathcal{O}_E(-1) \oplus \mathcal{O}_E(-2) \oplus \mathcal{O}_E(-7) \rightarrow (M/Mg)^\sim \rightarrow 0$$

Hence

$$X_{[M]} = \begin{pmatrix} \overline{m}_{11}^\sigma & \overline{m}_{12}^\sigma & \overline{m}_{13}^\sigma \\ \overline{m}_{21}^{\sigma^2} & \overline{m}_{22}^{\sigma^2} & \overline{m}_{23}^{\sigma^2} \\ 0 & 0 & \overline{m}_{33}^{\sigma^7} \end{pmatrix}$$

Therefore

$$s_{[M]} = \det X_{[M]} = \overline{m}_{11}^\sigma \otimes \overline{m}_{22}^{\sigma^2} \otimes \overline{m}_{33}^{\sigma^7} - \overline{m}_{12}^\sigma \otimes \overline{m}_{21}^{\sigma^2} \otimes \overline{m}_{33}^{\sigma^7}$$

which is a global section of the line bundle $\mathcal{L}^{p_M(\sigma)}$ where $p_M(t) = q_M(t)/(1-t) = t + 2t^2 + t^3 + t^4 + t^5 + t^6 + t^7$, see §1.4.

In general, the following result was shown in [1, 2].

Proposition 2.5. *Let M be a Cohen-Macaulay curve A -module, say with a minimal resolution*

$$0 \rightarrow \bigoplus_i A(-i)^{b_i} \xrightarrow{[M]} \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

Then the following holds.

- (1) *Up to a scalar multiple, $s_{[M]}$ is nonzero and independent of the choice of a minimal resolution for M .*
- (2) *For any integer l we have $s_{[M(l)]} = \sigma^l s_{[M]}$.*
- (3) *$s_{[M]} \in H^0(E, \mathcal{L}^{p_M(\sigma)})$ and $\deg \mathcal{L}^{p_M(\sigma)} = r_A \epsilon_M$.*
- (4) *The divisor of zeros of $s_{[M]}$ coincides with the divisor $\text{Div}(M)$.*

2.4. Further properties of divisors of curve modules. By Proposition 2.5, the divisor of a Cohen-Macaulay curve A -module M is the divisor of a section of the line bundle $\mathcal{L}^{p_M(\sigma)}$, and this line bundle depends only on the Hilbert series of M . This yields (see also [2])

Proposition 2.6. (1) *Let M, M' be two Cohen-Macaulay curve modules with the same Hilbert series $h_M(t) = h_{M'}(t)$. Then $\text{Div}(M) \sim \text{Div}(M')$.*

- (2) *Let D be a divisor on E . Then, for any Laurent power series $h(t) \in \mathbb{Z}((t))$ there is at most one $q \in E$ such that $D + (q) = \text{Div}(M)$ for some Cohen-Macaulay curve module M with Hilbert series $h(t)$.*

Proof. (1) As $h_M(t) = h_{M'}(t)$ we also have $p_M(t) = p_{M'}(t)$. Proposition 2.5 implies that $\text{Div}(M)$ and $\text{Div}(M')$ are both divisors of global sections of the same line bundle $\mathcal{L}^{p_M(\sigma)}$. Hence $\text{Div}(M)$ and $\text{Div}(M')$ are linearly equivalent.

(2) For the second statement, assume $q, q' \in E$ for which

$$D + (q) = \text{Div}(M), \quad D + (q') = \text{Div}(M')$$

for some Cohen-Macaulay curve modules M, M' with Hilbert series $h(t)$. By the first part of the current proposition and Proposition 2.1, $\text{Div}(M)$ and $\text{Div}(M')$ have the same sum in the group law of E . But this implies $q = q'$, ending the proof. \square

Remark 2.7. In Theorem 3.5 we prove a converse of Proposition 2.6(2). As mentioned in the introduction, this will be our key result to prove Theorem A.

In case of Cohen-Macaulay curve A -modules of the form $M = A/aA$ we have a more detailed version.

Lemma 2.8. *Let $n > 0$ be an integer and D a multiplicity-free effective divisor of degree $\leq r_{An}$. Then*

$$\mathcal{D} = \{b \in B_n \mid \text{Supp}(D) \subset \text{Supp}(\text{Div}(b))\}$$

is a k -linear subspace of B_n of dimension

$$\dim_k \mathcal{D} \begin{cases} = r_{An} - \deg D & \text{if } \deg D \leq r_{An} - 1 \\ \leq 1 & \text{if } \deg D = r_{An} \end{cases}$$

Proof. This follows from the category equivalence Γ_* and Riemann-Roch on the elliptic curve E . \square

From the previous lemma it is clear that, given $r_{An} - 2$ points on E , there are infinitely many sections on \mathcal{L}_n vanishing in these points. We will need a somewhat more refined version of this.

Lemma 2.9. *Let $n > 0$ be an integer, $q_1, \dots, q_{r_{An}-2} \in E$ be different points and \mathcal{Q} a finite set of points of E . There exists a homogeneous form $b \in B_n$ for which $\text{Div}(b)$ is multiplicity-free, $q_i \in \text{Supp Div}(b)$ and $\text{Supp Div}(b) \cap \mathcal{Q} = \emptyset$.*

2.5. Division in B . The following lemma is a useful criterion for division in the twisted homogeneous coordinate ring $B = \Gamma_*(\mathcal{O}_E)$.

Lemma 2.10. *Let $b \in B_n$ and $\tilde{b} \in B_m$ be nonzero. Then*

$$\text{Div}(b) = \text{Div}(\tilde{b}) + D \text{ for some effective divisor } D \Leftrightarrow b = \tilde{b}c \text{ for some } c \in B_{n-m}.$$

Proof. Recall that $B_n = H^0(E, \mathcal{L}_n)$ where \mathcal{L}_n is the invertible sheaf

$$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} = \mathcal{O}_E(n)$$

Write $\mathcal{L}_n = \mathcal{O}_E(D_n)$ for some divisor D_n on E . Using this notation, $\text{Div}(b) \sim D_n$ and $\text{Div}(\tilde{b}) \sim D_m$. It follows that $D \sim D_n - D_m$ where

$$\mathcal{O}_E(D_n - D_m) \cong \mathcal{L}_n \otimes \mathcal{L}_m^{-1} = \mathcal{L}^{\sigma^m} \otimes \mathcal{L}^{\sigma^{m+1}} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} = \mathcal{L}_{n-m}^{\sigma^m}$$

As D is effective there is a $c' \in H^0(E, \mathcal{L}_{n-m}^{\sigma^m})$ for which $\text{Div}(c') = D$. Thus

$$\text{Div}(b) = \text{Div}(\tilde{b}) + \text{Div}(c') = \text{Div}(\tilde{b} \otimes c') = \text{Div}(\tilde{b}c'^{\sigma^{-m}})$$

By [13, II Proposition 7.7] we have $b = \lambda \tilde{b}c'^{\sigma^{-m}}$ for some $0 \neq \lambda \in k$. Putting $c = \lambda c'^{\sigma^{-m}} \in H^0(E, \mathcal{L}_{n-m}) = B_{n-m}$ proves what we want. \square

From the previous lemma we deduce

Lemma 2.11. *Let $b, \tilde{b} \in B_n$ be nonzero. Assume $\text{Div}(b) = \text{Div}(\tilde{b}) - (p) + (q)$ for some $p, q \in E$. Then $p = q$ and $b = \tilde{b}c$ for some $c \in k$.*

Proof. By Proposition 2.6, $\text{Div}(b) \sim \text{Div}(\tilde{b})$. As these divisors have the same degree, Proposition 2.1 gives they have the same sum for the group law of A . Thus $p = q$. Invoking Lemma 2.10 completes the proof. \square

2.6. Quantum-irreducible divisors on E . Let M be a curve A -module. In [2] the author found a sufficient condition on $\text{Div}(M)$ for M to be critical. We will need this result. For convenience we briefly recall his treatment.

As Div is additive on short exact sequences (Proposition 2.2) we have

Lemma 2.12. ([2, Lemma 3.3]) *Let M be a curve A -module. If M is not critical then*

$$\text{Div}(M) = \text{Div}(M_1) + \text{Div}(M_2)$$

for some curve A -modules M_1, M_2 .

Inspired by the previous lemma, we say that an effective divisor D on E is *quantum-reducible* [2] if

$$D = \text{Div}(M) + D'$$

where M is a curve A -module and D' is an effective divisor of degree > 0 . We say D is *quantum-irreducible* if D is not quantum-reducible. By Proposition 2.2, any effective divisor of degree $< r_A$ is quantum-irreducible. We have

Lemma 2.13. *Let M be a curve A -module. Assume $\text{Div}(M) = D + (q)$ for some quantum-irreducible divisor D and $q \in E$. Then M is critical.*

Proof. Assume by contradiction that $\text{Div}(M)$ is not critical. By Lemma 2.12 we have $\text{Div}(M) = \text{Div}(M_1) + \text{Div}(M_2)$ for some curve A -modules M_1, M_2 . Since $\text{Div}(M) = D + (q)$ we must have $q \in \text{Supp } M_i$ for some $i = 1, 2$, say for $i = 2$. Then $D' = \text{Div}(M_2) - (q)$ is effective and of degree > 0 by Proposition 2.2(1). Now $D = \text{Div}(M_1) + D'$ contradicts the irreducibility of D . \square

The existence of quantum-irreducible divisors follows from (it is straightforward to extend the proof for cubic A)

Theorem 2.14. ([2, Theorem 3.7]) *For any positive integer n there exists a multiplicity-free quantum-irreducible effective divisor D of degree n on E , which is not the divisor of a curve A -module.*

Sketch of the proof. It is sufficient to construct a multiplicity-free effective divisor D which is not of the form $D = D' + D''$ for some effective divisors D', D'' where $D' \sim \text{Div } M$ for some critical curve module $M \in \text{grmod}(A)$. By Proposition 2.5, it is sufficient to exclude those D' for which $\mathcal{O}_E(D') \cong \mathcal{L}^{\sigma^n p_M(\sigma)}$ for some critical normalized curve A -module M and integer n . By [1] there are only finitely many possibilities for such $p_M(t) \in \mathbb{Z}[t, t^{-1}]$, as there are only finitely many possibilities for the Hilbert series for M . This is also part of Theorem B (a part for which we do not rely on the current theorem). Thus we have to exclude a countable number of divisors. As $k = \mathbb{C}$ is uncountable, we are finished. \square

3. PROOF OF THEOREM A

3.1. A set of equivalent conditions. Analogous to [12] we need equivalent versions of the conditions in Theorem A. The obvious proofs are left to the reader.

Lemma 3.1. *Let $(a_i), (b_i)$ be finitely supported sequences of integers, both not identically zero, and put $q_i = a_i - b_i$. The following sets of conditions are equivalent.*

- (1) Let q_μ be the lowest non-zero q_i and q_ν the highest non-zero q_i .
 - (a) $a_l = 0$ for $l < \mu$ and $l \geq \nu$.
 - (b) $a_\mu = q_\mu > 0$.
 - (c) $\sum_i q_i = 0$
 - (d) $\max(q_l, 0) \leq a_l \leq \sum_{i \leq l} q_i$ for all integers l .
- (2) Let a_μ be the lowest non-zero a_i and b_ν the highest non-zero b_i .
 - (a) The $(a_i), (b_i)$ are non-negative.
 - (b) $a_l = 0$ for $l \geq \nu$, $b_l = 0$ for $l \leq \mu$.
 - (c) $\sum_i a_i = \sum_i b_i$.
 - (d) $\sum_{i \leq l} b_i \leq \sum_{i < l} a_i$ for all integers l .
- (3) Put $m = \sum_i a_i$, $n = \sum_i b_i$.
 - (a) The $(a_i), (b_i)$ are non-negative.
 - (b) $m = n$.
 - (c) $\forall (\alpha, \beta) \in R_{m,n} : \beta \geq \alpha \Rightarrow (\alpha, \beta) \in L_{a,b}$.

Lemma 3.2. *Let $(a_i), (b_i)$ be finitely supported sequences of integers, both not identically zero, and put $q_i = a_i - b_i$. The following sets of conditions are equivalent.*

- (1) Let q_μ be the lowest non-zero q_i and q_ν the highest non-zero q_i .
 - (a) $a_l = 0$ for $l < \mu$ and $l \geq \nu$.
 - (b) $a_\mu = q_\mu > 0$.
 - (c) $\sum_i q_i = 0$
 - (d) $\max(q_l, 0) \leq a_l < \sum_{i \leq l} q_i$ for $\mu < l < \nu$.
 - (e) If A is cubic it is not true that $(a_\mu \geq 2$ and $\mu = \nu - 1)$.
- (2) Let a_μ be the lowest non-zero a_i and b_ν the highest non-zero b_i .
 - (a) The $(a_i), (b_i)$ are non-negative.
 - (b) $a_l = 0$ for $l \geq \nu$, $b_l = 0$ for $l \leq \mu$.
 - (c) $\sum_i a_i = \sum_i b_i$.
 - (d) $\sum_{i \leq l} b_i < \sum_{i < l} a_i$ for $\mu < l < \nu$.
 - (e) If A is cubic it is not true that $(n \geq 2$ and $\mu = \nu - 1)$.
- (3) Put $m = \sum_i a_i$, $n = \sum_i b_i$.
 - (a) The $(a_i), (b_i)$ are non-negative.
 - (b) $m = n$.
 - (c) $\forall (\alpha, \beta) \in R_{m,n} : \beta \geq \alpha - 1 \Rightarrow (\alpha, \beta) \in L_{a,b}$.
 - (d) If A is cubic it is not true that $(n \geq 2$ and $\forall \alpha, \beta : S(b)_\beta - S(a)_\alpha = 1)$.

3.2. Proof that the conditions in Theorem A are necessary. This was proved in [1] for quadratic A , and it is easy to extend it for cubic A . As the notations in [1] are quite different as in this current paper, we recall the arguments.

3.2.1. Proof that the conditions in Theorem A(1) are necessary. We will show that the equivalent conditions given in Lemma 3.1(2) are necessary.

Assume that $M \in \text{grmod}(A)$ is Cohen-Macaulay of GK-dimension two. Consider a minimal projective resolution of M

$$(3.1) \quad 0 \rightarrow \bigoplus_i A(-i)^{b_i} \xrightarrow{\phi} \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

There is nothing to prove for (2a) and expressing that M has rank zero gives (2c), so we discuss (2b) and (2d). The resolution (3.1) contains, for all integers l , a subcomplex of the form

$$(3.2) \quad \bigoplus_{i \leq l} A(-i)^{b_i} \xrightarrow{\phi_l} \bigoplus_{i \leq l} A(-i)^{a_i}$$

Since (3.1) is minimal all nonzero entries of a matrix representing ϕ have positive degree. Hence the image of $\bigoplus_{i \leq l} A(-i)^{b_i}$ under ϕ_l is contained in $\bigoplus_{i < l} A(-i)^{a_i}$. The fact that ϕ_l must be injective implies

$$(3.3) \quad \sum_{i \leq l} b_i \leq \sum_{i < l} a_i$$

from which we obtain (2d). In particular, if we take $l = \mu$ this shows that $b_i = 0$ for $i \leq \mu$. In order to prove that $a_i = 0$ for $i \geq \nu$, add $\sum_{i > l} (a_i - b_i)$ on both sides of (3.3) and use (2c) to obtain

$$a_l + \sum_{i > l} (a_i - b_i) \leq \sum_i (a_i - b_i) = 0$$

thus

$$(3.4) \quad \sum_{l \leq i} a_i \leq \sum_{l < i} b_i$$

Taking $l = \nu$ gives $a_i = 0$ for $i \geq \nu$.

This completes the proof that the conditions in Theorem A(1) are necessary.

3.2.2. Proof that the conditions in Theorem A(2) are necessary. We will show that the equivalent conditions given in Lemma 3.2(2) are necessary.

Let M be a critical Cohen-Macaulay module of GK-dimension two. Same reasoning as in §3.2.1 shows Lemma 3.2(2)(a-c). So we need to show that Lemma 3.2(2)(d-e) holds.

We will start with the proof of Lemma 3.2(2)(d), i.e.

$$\sum_{i \leq l} b_i < \sum_{i < l} a_i \text{ for } \mu < l < \nu$$

So assume by contradiction that there is some integer l where $\mu < l < \nu$ such that $\sum_{i \leq l} b_i = \sum_{i < l} a_i$. This means that, for the injective map (3.2), $\text{coker } \phi_l$ has GK-dimension ≤ 2 and is different from zero.

Note $\bigoplus_{i < l} A(-i)^{a_i}$ is not zero since $l > \mu$. We have a map $\text{coker } \phi_l \rightarrow M$ which we claim to be nonzero. Indeed, if this were the zero map then $\bigoplus_{i < l} A(-i)^{a_i} \rightarrow M$ is the zero map, which contradicts the minimality of the resolution (3.1). Hence $\text{coker } \phi_l \rightarrow M$ is nonzero. From this we get $\text{GKdim}(\text{coker } \phi_l) \geq 2$. Thus $\text{GKdim}(\text{coker } \phi_l) = 2$.

We will compare the multiplicity e_l of $\text{coker } \phi_l$ with the multiplicity e_M of M . As in the introduction, put $\epsilon_M = \iota_A e_M$ and $\epsilon_l = \iota_A e_l$. By (1.1) and (1.3) we have

$$\epsilon_M = \sum_i i(b_i - a_i) \text{ and } \epsilon_l = \sum_{i < l} i(b_i - a_i) + lb_l$$

We claim that $lb_l < \sum_{i \leq l} i(b_i - a_i)$. Indeed, this follows from

$$\begin{aligned} \sum_{l \leq i} i(b_i - a_i) &= l \sum_{l \leq i} (b_i - a_i) + \sum_{l+1 \leq i} (b_i - a_i) + \sum_{l+2 \leq i} (b_i - a_i) + \dots \\ &\geq lb_l + b_{l+1} + b_{l+2} + \dots \\ &> lb_l \end{aligned}$$

where the first inequality follows from (3.4) and the second one from the assumption that $l < \nu$. Thus we obtain

$$\epsilon_l < \sum_{i < l} i(b_i - a_i) + \sum_{l \leq i} i(b_i - a_i) = \epsilon_M$$

This means that $\text{coker } \phi_l$ has lower multiplicity than M . Hence the induced map $\text{coker } \phi_l \rightarrow M$ must be zero since M is assumed to be critical. But, as pointed out above, this implies that $\bigoplus_{i < l} A(-i)^{a_i} \rightarrow M$ is the zero map, which is impossible. This proves Lemma 3.2(2d).

What is left to prove is that Lemma 3.2(2e) holds. If, by contradiction, Lemma 3.2(2e) is not true then A is cubic and M admits a minimal resolution of the form

$$0 \rightarrow A(-\nu)^n \rightarrow A(-(\nu-1))^n \rightarrow M \rightarrow 0$$

By shift of grading, we may assume $\nu = 1$. We present the proof for $n = 2$. The arguments are easily extended for all $n \geq 2$. This will complete the proof that the conditions in Theorem A(2) are necessary.

Example 3.3. Assume A is cubic and M is an A -module admitting a minimal resolution of the form

$$(3.5) \quad 0 \rightarrow A(-1)^2 \xrightarrow{\begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}} A^2 \rightarrow M \rightarrow 0$$

where the entries $l_i = \alpha_i x + \beta_i y \in A_1$ are linear forms ($\alpha_i, \beta_i \in k$). Since

$$h_M(t) = h_A(t)(2 - 2t) = \frac{2}{(1-t)^2(1+t)} = 2 + 2t + 4t^2 + 4t^3 + 6t^4 + \dots$$

we have $\text{GKdim } M = 2$, $e_M = 1$ and $\epsilon_M = 2$. We will show that M is not critical. Let $(x_0, y_0) \in \mathbb{P}^1$ be a solution of the quadratic equation

$$\det \begin{pmatrix} \alpha_1 x_0 + \beta_1 y_0 & \alpha_2 x_0 + \beta_2 y_0 \\ \alpha_3 x_0 + \beta_3 y_0 & \alpha_4 x_0 + \beta_4 y_0 \end{pmatrix} = 0$$

Thus there is a nonzero $(\lambda, \mu) \in k^2$ for which

$$\begin{pmatrix} \alpha_1 x_0 + \beta_1 y_0 & \alpha_2 x_0 + \beta_2 y_0 \\ \alpha_3 x_0 + \beta_3 y_0 & \alpha_4 x_0 + \beta_4 y_0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0$$

Consider the linear form $l = y_0 x - x_0 y \in A_1$. Up to scalar multiplication, l is the unique linear form $\alpha x + \beta y$ for which $\alpha x_0 + \beta y_0 = 0$. This means that

$$\begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} l$$

for some $\gamma, \delta \in k$. Note that $(\gamma, \delta) \neq (0, 0)$ since (3.5) is exact. This leads to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A(-1) & \xrightarrow{l} & A & \longrightarrow & A/LA \longrightarrow 0 \\ & & \left(\begin{array}{c} \lambda \\ \mu \end{array}\right) \downarrow & & \left(\begin{array}{cc} l_1 & l_2 \\ l_3 & l_4 \end{array}\right) \downarrow & & \left(\begin{array}{c} \gamma \\ \delta \end{array}\right) \downarrow \\ 0 & \longrightarrow & A(-1)^2 & \longrightarrow & A^2 & \longrightarrow & M \longrightarrow 0 \end{array}$$

Hence there is a nonzero map $A/LA \rightarrow M$. As A/LA has multiplicity $1/2$ and M has multiplicity 1, this shows that M is not critical, a contradiction.

3.3. Proof that the conditions in Theorem A(1) are sufficient. We fix finitely supported sequences $(a_i), (b_i)$ of non-negative integers such that $\sum_i a_i = \sum_i b_i = n$ and we assume the ladder condition holds:

$$(3.6) \quad \forall(\alpha, \beta) \in R : \beta \geq \alpha \Rightarrow (\alpha, \beta) \in L_{a,b}$$

Thus $S(b)_\alpha - S(a)_\alpha > 0$ for $1 \leq \alpha \leq n$. Pick nonzero homogeneous elements $h_{\alpha\alpha} \in B_{S(b)_\alpha - S(a)_\alpha}$. Let $h_{\alpha\alpha}$ denote also any pre-image of $h_{\alpha\alpha}$ in A . As A is a domain, multiplication by $h_{\alpha\alpha}$ is injective. Let H_α be the corresponding cokernels

$$0 \rightarrow A(-S(b)_\alpha) \xrightarrow{h_{\alpha\alpha}} A(-S(a)_\alpha) \rightarrow H_\alpha \rightarrow 0$$

for $1 \leq \alpha \leq n$. Then A -module $M = H_1 \oplus \cdots \oplus H_n$ admits a minimal resolution

$$0 \rightarrow \bigoplus_i A(-i)^{b_i} \xrightarrow{N} \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

where

$$N = \begin{pmatrix} h_{11} & 0 & \dots & 0 \\ 0 & h_{22} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & h_{nn} \end{pmatrix}$$

Hence M has projective dimension one, with graded Betti numbers $(a_i), (b_i)$. That M has GK-dimension two is easy to see (see also the proof of Lemma 3.4(3) below). As we have chosen $h_{\alpha\alpha} \in B$, the cyclic modules H_α are g -torsionfree. Hence M is also g -torsionfree. This completes the proof.

3.4. Proof that the conditions in Theorem A(2) are sufficient. We fix finitely supported sequences $(a_i), (b_i)$ of non-negative integers for which $\sum_i a_i = \sum_i b_i = n$ and we assume that the ladder condition

$$\forall(\alpha, \beta) \in R : \beta \geq \alpha - 1 \Rightarrow (\alpha, \beta) \in L_{a,b}$$

is true, together with condition Theorem A(2d):

If A is cubic it is not true that $(n \geq 2$ and $\forall \alpha, \beta : S(b)_\beta - S(a)_\alpha = 1)$.

Put $\epsilon = \sum_i i(b_i - a_i)$. We are motivated by the following

Lemma 3.4. *Assume that we have a map $N : \bigoplus_i \mathcal{O}_E(-i)^{b_i} \rightarrow \bigoplus_i \mathcal{O}_E(-i)^{a_i}$ such that $X_N(p)$ has maximal rank for all but finitely many points $p \in E$. Then*

(1) *N is injective, i.e. we have a short exact sequence in $\text{coh}(E)$*

$$0 \rightarrow \bigoplus_i \mathcal{O}_E(-i)^{b_i} \xrightarrow{N} \bigoplus_i \mathcal{O}_E(-i)^{a_i} \rightarrow \mathcal{N} \rightarrow 0$$

where $\mathcal{N} \in \text{coh}(E)$ has finite length.

(2) Applying Γ_* to N induces a short exact sequence in $\text{grmod}(B)$

$$0 \rightarrow \bigoplus_i B(-i)^{b_i} \xrightarrow{\Gamma_*(N)} \bigoplus_i B(-i)^{a_i} \rightarrow M' \rightarrow 0$$

where $M' \in \text{grmod}(B)$ is pure of GK-dimension one and $\widetilde{M}' = \mathcal{N}$.

(3) Arbitrary lifting of $\Gamma_*(N)$ under the surjective map

$$\text{Hom}_A(\bigoplus_i A(-i)^{b_i}, \bigoplus_i A(-i)^{a_i}) \rightarrow \text{Hom}_B(\bigoplus_i B(-i)^{b_i}, \bigoplus_i B(-i)^{a_i}) : f \mapsto f \otimes_A B$$

yields a short exact sequence in $\text{grmod}(A)$

$$0 \rightarrow \bigoplus_i A(-i)^{b_i} \rightarrow \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

where $M \in \text{grmod}(A)$ has GK-dimension two and $\epsilon_M = \sum_i i(b_i - a_i)$. Moreover, $M/Mg = M'$ and M is g -torsionfree.

Proof. (1) If N were not injective then the kernel of N would, as a subsheaf of the vector bundle $\bigoplus_i \mathcal{O}_E(-i)^{b_i}$, have rank > 0 . Since $\sum_i a_i = \sum_i b_i$ the same is true for the cokernel of N . Then $\text{coker } N$ would not be supported on finitely many points in E . But this means that $X_N(p)$ has non-maximal rank for infinitely many points $p \in E$, a contradiction. Thus N is injective. That $\mathcal{N} = \text{coker } N$ has finite length follows from $\sum_i a_i = \sum_i b_i$.

(2) Apply the functor $\Gamma_* = \bigoplus_{m \geq 0} H^0(E, - \otimes_E \mathcal{O}_E(m))$ to the short exact sequence in (1) and use $\Gamma_*(\mathcal{O}(l)) = B(l)$ for all integers l . As Γ_* is left exact, $\Gamma_*(N)$ is injective. Since $\underline{\text{Ext}}_B^1(k, B) = 0$, M' contains no finite dimensional submodules. Hence M' is pure. Application of the exact functor $\widetilde{(-)}$ shows $\widetilde{M}' = \mathcal{N}$ and $\text{GKdim } M' = 1$.

(3) Let f denote such an arbitrary lifting. Thus we have a map in $\text{grmod}(A)$

$$\bigoplus_i A(-i)^{b_i} \xrightarrow{f} \bigoplus_i A(-i)^{a_i}$$

and $f \otimes_A B = \Gamma_*(N)$. Representing f by left multiplication by a matrix $[M]$ then $\Gamma_*(N)$ is represented by left multiplication by $[\overline{M}]$, the matrix obtained from $[M]$ by taking the image of the entries in B . For any $a \in \ker f$ the equation $[M] \cdot a = 0$ in $\text{grmod}(A)$ yields $[\overline{M}] \cdot \overline{a} = 0$ in $\text{grmod}(B)$ hence $\overline{a} = 0$ by the injectivity of $\Gamma_*(N)$. Thus $(\ker f) \otimes_A B = 0$ i.e. $\ker f / (\ker f) \cdot g = 0$. Hence if $\ker f$ were nonzero then $\text{GKdim}(\ker f) \leq 1$, see [11, Lemma 2.8.2]. But this is impossible by the pureness of $\bigoplus_i A(-i)^{b_i}$. Hence f is injective.

Write $M = \text{coker } f$. Applying $- \otimes_A B$ to the short exact sequence

$$0 \rightarrow \bigoplus_i A(-i)^{b_i} \xrightarrow{f} \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

and using $f \otimes_A B = \Gamma_*(N)$, we get $M/Mg = M'$ and $\text{Tor}_1^A(M, B) = 0$. Thus M is g -torsionfree.

What remains to prove is $\text{GKdim } M = 2$. By (1.3) and Lemma 3.1(2c) we have

$$\lim_{t \rightarrow 0} (1-t)^3 h_M(t) = \sum_i (a_i - b_i) = 0$$

thus $\text{GKdim } M < 3$, and

$$\begin{aligned} 2 \lim_{t \rightarrow 0} (1-t)^2 h_M(t) &= \sum_i i(b_i - a_i) = \sum_i (\nu + 1 - i)(a_i - b_i) \\ &= \sum_{i \leq \mu} (a_i - b_i) + \sum_{i \leq \mu+1} (a_i - b_i) + \sum_{i \leq \mu+2} (a_i - b_i) + \dots \\ &\geq \sum_i a_i \end{aligned}$$

where we have used Lemma 3.1(2d) to obtain the inequality. Since $\sum_i a_i > 0$ this proves $\lim_{t \rightarrow 0} (1-t)^2 h_M(t) > 0$ i.e. $\text{GKdim } M = 2$, and $\epsilon_M = \sum_i i(b_i - a_i)$. This completes the proof. \square

Our proof that the conditions in Theorem A(2) are sufficient follows from the following stronger result.

Theorem 3.5. *Let ϵ be a positive integer and let D be a multiplicity-free effective divisor of degree $r_A \epsilon - 1$. Then there exists a g -torsionfree module $M \in \text{grmod}(A)$ of GK-dimension two and projective dimension one which has graded Betti-numbers $(a_i), (b_i)$ i.e. M admits a minimal resolution of the form*

$$0 \rightarrow \bigoplus_i A(-i)^{b_i} \rightarrow \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

and for which $\text{Div}(M) = D + (q)$ for some $q \in E$.

Indeed, for then we choose a multiplicity-free quantum-irreducible effective divisor D of degree $r_A \epsilon - 1$, whose existence is asserted from Theorem 2.14. Lemma 2.13 implies that the module M in Theorem 3.5 is critical.

Thus in order to complete the proof of Theorem A(2) it will be sufficient to prove Theorem 3.5. This will be done below.

Proof of Theorem 3.5. Throughout the proof we fix a multiplicity-free effective divisor $D = (q_1) + (q_2) + \dots + (q_{r_A \epsilon - 1})$ of degree $r_A \epsilon - 1$. As in Lemma 3.2, let a_μ be the lowest non-zero a_i and b_ν be the highest non-zero b_i . Thus $\mu = S(a)_1$ and $\nu = S(b)_n$. Write $u = \sum_{i < \nu} b_i$ and $v = b_\nu - 1$.

We break up the proof into six steps.

Step 1. Our first step in the proof is to choose a particular $n \times (n-1)$ matrix of the form (only the nonzero entries are indicated)

$$H = [H_U \mid H_V] = \left(\begin{array}{cccc|cccc} h_{11} & h_{12} & \dots & h_{1u} & h_{1,u+1} & \dots & h_{1,n-1} \\ h_{21} & & & & & & \\ & h_{32} & & & & & \\ & & \ddots & & & & \\ & & & h_{u+1,u} & & & \\ & & & & h_{u+2,u+1} & & \\ & & & & & \ddots & \\ & & & & & & h_{n,n-1} \end{array} \right)$$

whose entries are homogeneous forms $h_{1\beta} \in B_{S(b)_\beta - S(a)_1}$, $h_{\beta+1,\beta} \in B_{S(b)_\beta - S(a)_{\beta+1}}$ satisfying the following conditions:

- (*) The divisors $\text{Div } h_{\alpha\beta}^{S(a)\alpha}$ of the (nonzero) entries $h_{\alpha\beta}$ in H_U are multiplicity-free and have pairwise disjoint support.
- (**) The support of the divisor $\text{Div } h_{\alpha\beta}^{S(a)\alpha}$ of any entry $h_{\alpha\beta}$ in H_U is disjoint with the support of the divisor $\text{Div } h_{\alpha'\beta'}^{S(a)\alpha'}$ of any entry $h_{\alpha'\beta'}$ in H_V .
- (***) The divisors $\text{Div } h_{\alpha'\beta'}^{S(a)\alpha'}$ of the entries $h_{\alpha'\beta'}$ in H_V are multiplicity-free. They have pairwise disjoint support unless they appear in the same column $\beta' - u$ of H_V . In that case, $\text{Supp}(\text{Div } h_{1,\beta'}^{S(a)1}) \cap \text{Supp}(\text{Div } h_{\beta'+1,\beta'}^{S(a)\beta'+1}) = \{q_{\beta'-u}\}$.

Observe that, due to Lemma 2.9, it is possible to choose such matrices H_U, H_V except in the following situation:

(3.7) A is cubic, $v > 0$ and two linear forms appear in the same column of H_V

This is because for any two linear forms in A (where A is cubic) their divisors have either disjoint support or the same support (being two distinct points). However, by (1.4) it is easy to see that (3.7) is same as saying that $n \geq 2$ and all entries of H are linear forms, i.e. $\mu = \nu - 1$. By Lemma 3.2, this is exactly excluded by condition Theorem A(2d)! In other words, (3.7) does not occur.

Step 2. By construction, the matrix H in Step 1 represents a map

$$H : \bigoplus_{i < \nu} \mathcal{O}_E(-i)^{b_i} \oplus \mathcal{O}_E(-\nu)^{b_\nu-1} \rightarrow \bigoplus_i \mathcal{O}_E(-i)^{a_i}$$

Recall §2.2 that in this case the matrix X_H is given by

$$X_H = \left(\begin{array}{cccc|cccc} h_{11}^{\sigma^{S(a)1}} & h_{12}^{\sigma^{S(a)1}} & \dots & h_{1u}^{\sigma^{S(a)1}} & h_{1,u+1}^{\sigma^{S(a)1}} & \dots & h_{1,n-1}^{\sigma^{S(a)1}} & \\ h_{21}^{\sigma^{S(a)2}} & & & & & & & \\ & h_{32}^{\sigma^{S(a)3}} & & & & & & \\ & & \ddots & & & & & \\ & & & h_{u+1,u}^{\sigma^{S(a)u+1}} & & & & \\ & & & & h_{u+2,u+1}^{\sigma^{S(a)u+2}} & & & \\ & & & & & \ddots & & \\ & & & & & & h_{n,n-1}^{\sigma^{S(a)n}} & \end{array} \right)$$

Therefore, by Step 1 we find

$$\text{rank } X_H(p) = \begin{cases} n-1 & \text{if } p \in E \setminus \{q_1, \dots, q_v\} \\ n-2 & \text{if } p \in \{q_1, \dots, q_v\} \end{cases}$$

Step 3. Any choice of homogeneous forms $d_\alpha \in B_{S(b)_n - S(a)_\alpha}$, $\alpha = 1, \dots, n$ determines a matrix

$$(3.8) \quad [H \mid d] = \left(\begin{array}{cccc|c} h_{11} & h_{12} & \dots & h_{1,n-1} & d_1 \\ h_{21} & & & & d_2 \\ & h_{32} & & & d_3 \\ & & \ddots & & \vdots \\ & & & h_{n,n-1} & d_n \end{array} \right)$$

which represents a map $[H \mid d] : \bigoplus_i \mathcal{O}_E(-i)^{b_i} \rightarrow \bigoplus_i \mathcal{O}_E(-i)^{a_i}$. We then consider the k -linear map

$$\theta : \bigoplus_{\alpha=1}^n B_{S(b)_n - S(a)_\alpha} \rightarrow H^0(E, \mathcal{L}^{p(\sigma)}) : d \mapsto \det X_{[H \mid d]}$$

where $p(t) = \sum_i (a_i - b_i)t^i / (1-t) \in \mathbb{Z}[t, t^{-1}]$. Furthermore, by Step 2 the image of θ is contained in the k -linear subspace

$$W = \{s \in H^0(E, \mathcal{L}^{p(\sigma)}) \mid s(q_i) = 0 \text{ for } i = 1, \dots, v\} \subset H^0(E, \mathcal{L}^{p(\sigma)})$$

We claim that $\text{im } \theta = W$. This will follow from the Steps 4 and 5 below.

Step 4. If $\dim_k \ker \theta = \sum_{\alpha=1}^{n-1} \dim_k B_{S(b)_n - S(b)_\alpha}$ then $\text{im } \theta = W$. Indeed, as D is multiplicity-free the conditions $s(q_i) = 0$ for $i = 1, \dots, v$ are k -linear independent, hence $\text{codim } W = v$. Thus

$$\begin{aligned} \dim_k W &= \dim_k H^0(E, \mathcal{L}^{p(\sigma)}) - v = \sum_{\alpha=1}^n \dim_k B_{S(b)_\alpha - S(a)_\alpha} - v \\ &= \sum_{\alpha=1}^n r_A(S(b)_\alpha - S(a)_\alpha) - v \end{aligned}$$

while on the other hand

$$\begin{aligned} \dim_k \text{im } \theta &= \sum_{\alpha=1}^n \dim_k B_{S(b)_n - S(a)_\alpha} - \dim_k \ker \theta \\ &= \sum_{\alpha=1}^n \dim_k B_{S(b)_n - S(a)_\alpha} - \sum_{\alpha=1}^{n-1} \dim_k B_{S(b)_n - S(b)_\alpha} \\ &= \sum_{\alpha=1}^n r_A(S(b)_n - S(a)_\alpha) - \sum_{\alpha=1}^n r_A(S(b)_n - S(b)_\alpha) - (b_\tau - 1) \\ &= \sum_{\alpha=1}^n r_A(S(b)_\alpha - S(a)_\alpha) - v \end{aligned}$$

Step 5. $\dim_k \ker \theta = \sum_{\alpha=1}^{n-1} \dim_k B_{S(b)_n - S(b)_\alpha}$. We prove this as follows. For any choice of homogeneous elements $c_\alpha \in B_{S(b)_n - S(b)_\alpha}$ for $\alpha = 1, \dots, n-1$, putting

$$(3.9) \quad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1,n-1} \\ h_{21} & & & \\ & h_{32} & & \\ & & \ddots & \\ & & & h_{n,n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix}$$

yields an element $d = (d_1, \dots, d_n) \in \bigoplus_{\alpha=1}^n B_{S(b)_n - S(a)_\alpha}$ in the kernel of θ . Thus we have a k -linear map

$$\tilde{\theta} : \bigoplus_{\alpha=1}^{n-1} B_{S(b)_n - S(b)_\alpha} \rightarrow \ker \theta : (c_1, \dots, c_{n-1})^t \mapsto H \cdot (c_1, \dots, c_{n-1})^t$$

which is injective by the fact that the entries of H are nonzero (Step 1) and B is a domain. Hence in order to prove Step 5 it suffices to show $\tilde{\theta}$ is surjective.

Pick $d = (d_1, \dots, d_n) \in \ker \theta$. By Step 2 we may solve (3.9) locally at $p \in E \setminus \{q_1, \dots, q_v\}$, i.e. we may find a solution $\lambda(p) = (\lambda_1(p), \dots, \lambda_{n-1}(p))$, where

$$\lambda_\alpha(p) \in \left(\mathcal{L}_{S(b)_n - S(b)_\alpha}^{\sigma^{S(b)_\alpha}} \right)_p / m_p \left(\mathcal{L}_{S(b)_n - S(b)_\alpha}^{\sigma^{S(b)_\alpha}} \right)_p$$

such that

$$(3.10) \quad \begin{pmatrix} d_1^{\sigma^{S(a)_1}}(p) \\ d_2^{\sigma^{S(a)_2}}(p) \\ d_3^{\sigma^{S(a)_3}}(p) \\ \vdots \\ d_n^{\sigma^{S(a)_n}}(p) \end{pmatrix} = \begin{pmatrix} h_{11}^{\sigma^{S(a)_1}}(p) & h_{12}^{\sigma^{S(a)_1}}(p) & \cdots & h_{1,n-1}^{\sigma^{S(a)_1}}(p) \\ h_{21}^{\sigma^{S(a)_2}}(p) & & & \\ & h_{32}^{\sigma^{S(a)_3}}(p) & & \\ & & \ddots & \\ & & & h_{n,n-1}^{\sigma^{S(a)_n}}(p) \end{pmatrix} \otimes \begin{pmatrix} \lambda_1(p) \\ \lambda_2(p) \\ \vdots \\ \lambda_{n-1}(p) \end{pmatrix}$$

To show that we can solve (3.9) globally, we proceed as follows.

- For $\beta = 1, \dots, u$ the $\beta + 1$ -th equation in (3.10) becomes

$$(3.11) \quad d_{\beta+1}(p^{\sigma^{S(a)_{\beta+1}}}) = h_{\beta+1,\beta}(p^{\sigma^{S(a)_{\beta+1}}}) \otimes \lambda_\beta(p) \text{ for all } p \in E \setminus \{q_1, \dots, q_v\}$$

By Step 1, $q_1, \dots, q_v \notin \text{Div } h_{\beta+1,\beta}^{\sigma^{S(a)_{\beta+1}}}$. Hence we deduce from (3.11)

$$d_{\beta+1} = 0 \text{ or } \text{Div } d_{\beta+1} = \text{Div } h_{\beta+1,\beta} + D'$$

for some effective divisor D' . By Lemma 2.10 this means that $d_{\beta+1} = h_{\beta+1,\beta} c_\beta$ for some $c_\beta \in B_{S(b)_n - S(b)_\beta}$.

- For $\beta = u + 1, \dots, n - 1$ the $\beta + 1$ -th equation in (3.10) becomes

$$(3.12) \quad d_{\beta+1}(p^{\sigma^{S(a)_{\beta+1}}}) = h_{\beta+1,\beta}(p^{\sigma^{S(a)_{\beta+1}}}) \otimes \lambda_\beta(p) \text{ for all } p \in E \setminus \{q_1, \dots, q_v\}$$

As $S(b)_\beta = S(b)_n$ we have $\deg d_{\beta+1} = \deg h_{\beta+1,\beta}$ (if $d_{\beta+1} \neq 0$). By Step 1 there is only one $i = 1, \dots, v$ for which $q_i^{\sigma^{S(a)_{\beta+1}}} \in \text{Div } h_{\beta+1,\beta}$. As $h_{\beta+1,\beta}^{\sigma^{S(a)_{\beta+1}}}$ is multiplicity-free Lemma 2.11 yields $d_{\beta+1} = h_{\beta+1,\beta} c_\beta$ for some $c_\beta \in B_{S(b)_n - S(b)_\beta} = k$.

- Finally, by the previous two items the first equation in (3.10) becomes

$$\begin{aligned} d_1(p^{\sigma^{S(a)_1}}) &= h_{11}(p^{\sigma^{S(a)_1}}) \otimes \lambda_1(p) + \cdots + h_{1,n-1}(p^{\sigma^{S(a)_1}}) \otimes \lambda_{n-1}(p) \\ &= (h_{11}c_1 + h_{12}c_2 + \cdots + h_{1,n-1}c_{n-1})(p^{\sigma^{S(a)_1}}) \end{aligned}$$

for $p \in E \setminus \{q_1, \dots, q_v\}$. Hence $(d_1 - \sum_{\beta=1}^{n-1} h_{1\beta} c_\beta)(p) = 0$ for all but finitely many $p \in E$. This clearly implies $d_1 = \sum_{\beta=1}^{n-1} h_{1\beta} c_\beta$.

It follows that d is of the form (3.9). We have shown that $\tilde{\theta}$ is surjective. This ends the proof of Step 5.

Step 6. As $\dim_k H^0(E, \mathcal{L}^{p(\sigma)}) = r_A \epsilon$, we may pick a global section $s \in H^0(E, \mathcal{L}^{p(\sigma)})$ for which $\text{Div}(s) = D + (q)$ for some $q \in E$. By construction we have $s \in W$. By Steps 4 and 5, we have $\text{im } \theta = W$. Thus we may find homogeneous forms $d_\alpha \in B_{S(b)_n - S(a)_\alpha}$, $\alpha = 1, \dots, n$ for which $\det X_{[H|d]} = s$. By Lemma 3.4, there is a short exact sequence in $\text{grmod}(A)$

$$0 \rightarrow \bigoplus_i A(-i)^{b_i} \xrightarrow{\Gamma_*([H|d])_A} \bigoplus_i A(-i)^{a_i} \rightarrow M \rightarrow 0$$

where M is g -torsionfree of GK-dimension two. By construction, $\text{Div}(M) = \text{Div}(s) = D + (p)$. This completes the proof of Theorem 3.5. \square

4. PROOF OF THEOREM B AND OTHER PROPERTIES OF HILBERT SERIES

Proof of Theorem B. First, let M be a normalized Cohen-Macaulay A -module of GK-dimension two and multiplicity e . Writing the Hilbert series $h_M(t)$ of M in the form (1.1) we see that there is a Laurent polynomial $s(t)$ for which

$$(4.1) \quad h_M(t) = h_A(t)(\epsilon(1-t) - s(t)(1-t)^2)$$

where $\epsilon = \iota_A e$. Since M is normalized we have $M_{<0} = 0$, thus $s(t) \in \mathbb{Z}[t]$.

Let $(a_i), (b_i)$ denote the graded Betti numbers of M and consider the characteristic polynomial $q_M(t) = \sum_i q_i t^i = \sum_i (a_i - b_i) t^i$. Then $q_M(t)/(1-t) = \sum_l p_l t^l$ where $p_l = \sum_{i \leq l} q_i$. By §3.3, the conditions of Lemma 3.1(1)(a-d) hold. Note that, as M is normal, a_0 is the lowest non-zero a_i i.e. $\mu = 0$. In particular,

$$p_l \begin{cases} > 0 & \text{for } l = 0 \\ \geq 0 & \text{for } 0 < l < \nu \\ = 0 & \text{else} \end{cases}$$

Combining (1.3) and (4.1) we have

$$(4.2) \quad s(t)(1-t) = \epsilon - \sum_l p_l t^l$$

Multiplying (4.2) by $1/(1-t) = 1 + t + t^2 + \dots$ shows that $s(t)$ is of the form

$$\epsilon > s_0 \geq s_1 \geq \dots \geq 0 \quad \text{where} \quad s_i = \epsilon - (p_0 + p_1 + \dots + p_i)$$

If M is in addition critical, Lemma 3.2(1d) implies $p_l > 0$ for $0 \leq l < \nu$. By same reasoning as above we find that $s(t)$ is of the form

$$\epsilon > s_0 > s_1 > \dots \geq 0$$

In case A is cubic, Lemma 3.2(1e) requires in addition that $q(t)$ is not of the form $n(1-t)$ for $n = \sum_i a_i = \sum_i b_i \geq 2$. This is the same as saying that in case $\epsilon \geq 2$ $q(t) \neq \epsilon(1-t)$. In other words, in case $\epsilon \geq 2$ we have $s(t) \neq 0$.

The converse of Theorem B follows by reversing the arguments. \square

Remark 4.1. From Theorem B we may deduce the following combinatorial result. For positive integers m, n let $p(D, n, < m)$ denote the number of partitions of n with distinct parts in which every part is strictly smaller than m . Needless to say that $p(D, n, < m) = 0$ for $n > m(m-1)/2$. Corollary 1.4 now yields

$$\sum_{n \geq 0} p(D, n, < m) = 2^{m-1}$$

for all positive integers m .

We also mention

Corollary 4.2. *Let $\epsilon > 0$ be an integer. The number of finitely supported sequences $(a_i), (b_i)$ which occur as the graded Betti numbers of a (resp. critical) normalized Cohen-Macaulay A -module M of GK-dimension two having Hilbert series*

$$h_M(t) = h_A(t)(\epsilon(1-t) - s(t)(1-t)^2)$$

is equal to

$$[1 + \min(\epsilon - s_0, s_0 - s_1)] \cdot \prod_{1 < l} [1 + \min(s_{l-2} - s_{l-1}, s_{l-1} - s_l)]$$

resp.

$$(4.3) \quad \min(\epsilon - s_0, s_0 - s_1) \cdot \prod_{1 < l} \min(s_{l-2} - s_{l-1}, s_{l-1} - s_l)$$

This number (4.3) is bigger than one if and only if there are two consecutive downward jumps of length ≥ 2 in the coefficients of $\epsilon t^{-1} + s(t)$.

Proof. The number of solutions to the conditions Lemma 3.1(1)(a-d) is

$$\prod_{\mu < l < \nu} \left(\left(\sum_{i \leq l} q_i \right) - \max(q_l, 0) + 1 \right) = \prod_{l > \mu} \min \left(1 + \sum_{i < l} q_i, 1 + \sum_{i \leq l} q_i \right)$$

Since we restrict to normalized modules we have $\mu = 0$. Noting that $q_0 = \epsilon - s_0$ and $\sum_{i \leq l} q_i = s_{l-1} - s_l$ for $l > 0$ yields that the number of solutions is equal to

$$\min(1 + \epsilon - s_0, 1 + s_0 - s_1) \cdot \prod_{1 < l} \min(1 + s_{l-2} - s_{l-1}, 1 + s_{l-1} - s_l)$$

Same reasoning in the critical case. \square

APPENDIX A. HILBERT SERIES UP TO $\epsilon = 4$

Let A be a generic three-dimensional Artin-Schelter regular algebra, either quadratic or cubic §1.2. Let M be a normalized critical Cohen-Macaulay graded right A -module of GK-dimension two. According to Theorem B the Hilbert series of M has the form





$$h_M(t) = h_A(t)(\epsilon(1-t) - s_M(t)(1-t)^2)$$





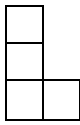
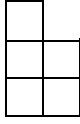
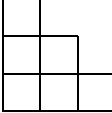
where $\epsilon > 0$ is an integer and $s_M(t) \in \mathbb{Z}[t]$ is a polynomial of the form

$$\epsilon > s_0 > s_1 > \dots \geq 0 \text{ and if } A \text{ is cubic and } \epsilon > 1 \text{ then } s(t) \neq 0$$

The multiplicity of M is given by $e_M = \epsilon/2$. For the cases $\epsilon \leq 4$ we list the possible Hilbert series for M , the corresponding $s(t)$ and the possible minimal resolutions of M . Recall that

$$r_A = \begin{cases} 3 & \text{if } A \text{ is quadratic} \\ 2 & \text{if } A \text{ is cubic} \end{cases}$$

$\epsilon = 1$	$h_M(t) = \begin{cases} 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \dots & \text{if } r_A = 3 \\ 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 0$ $0 \rightarrow A(-1) \rightarrow A \rightarrow M \rightarrow 0$
$\epsilon = 2$	$h_M(t) = \begin{cases} 2 + 4t + 6t^2 + 8t^3 + 10t^4 + 12t^5 + \dots & \text{if } r_A = 3 \\ \emptyset & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 0$ $0 \rightarrow A(-1)^2 \rightarrow A^2 \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 1 + 3t + 5t^2 + 7t^3 + 9t^4 + 11t^5 + \dots & \text{if } r_A = 3 \\ 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 1$ $0 \rightarrow A(-2) \rightarrow A \rightarrow M \rightarrow 0$
$\epsilon = 3$	$h_M(t) = \begin{cases} 3 + 6t + 9t^2 + 12t^3 + 15t^4 + 18t^5 + \dots & \text{if } r_A = 3 \\ \emptyset & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 0$ $0 \rightarrow A(-1)^3 \rightarrow A^3 \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 2 + 5t + 8t^2 + 11t^3 + 14t^4 + 17t^5 + \dots & \text{if } r_A = 3 \\ 2 + 3t + 5t^2 + 6t^3 + 8t^4 + 9t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 1$ $0 \rightarrow A(-1) \oplus A(-2) \rightarrow A^2 \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 1 + 4t + 7t^2 + 10t^3 + 13t^4 + 16t^5 + \dots & \text{if } r_A = 3 \\ 1 + 3t + 4t^2 + 6t^3 + 7t^4 + 9t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 2$ $0 \rightarrow A(-2)^2 \rightarrow A \oplus A(-1) \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 1 + 3t + 6t^2 + 9t^3 + 12t^4 + 15t^5 + \dots & \text{if } r_A = 3 \\ 1 + 2t + 4t^2 + 5t^3 + 7t^4 + 8t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 2 + t$ $0 \rightarrow A(-3) \rightarrow A \rightarrow M \rightarrow 0$

$\epsilon = 4$	$h_M(t) = \begin{cases} 4 + 8t + 12t^2 + 16t^3 + 20t^4 + 24t^5 + \dots & \text{if } r_A = 3 \\ \emptyset & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 0$ $0 \rightarrow A(-1)^4 \rightarrow A^4 \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 3 + 7t + 11t^2 + 15t^3 + 19t^4 + 23t^5 + \dots & \text{if } r_A = 3 \\ 3 + 4t + 7t^2 + 8t^3 + 11t^4 + 12t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 1$ $0 \rightarrow A(-1)^2 \oplus A(-2) \rightarrow A^3 \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 2 + 6t + 10t^2 + 14t^3 + 18t^4 + 22t^5 + \dots & \text{if } r_A = 3 \\ 2 + 4t + 6t^2 + 8t^3 + 10t^4 + 12t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 2$ $0 \rightarrow A(-2)^2 \rightarrow A^2 \rightarrow M \rightarrow 0$ $0 \rightarrow A(-1) \oplus A(-2)^2 \rightarrow A^2 \oplus A(-1) \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 2 + 5t + 9t^2 + 13t^3 + 17t^4 + 21t^5 + \dots & \text{if } r_A = 3 \\ 2 + 3t + 6t^2 + 7t^3 + 10t^4 + 11t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 2 + t$ $0 \rightarrow A(-1) \oplus A(-3) \rightarrow A^2 \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 1 + 5t + 9t^2 + 13t^3 + 17t^4 + 21t^5 + \dots & \text{if } r_A = 3 \\ 1 + 4t + 5t^2 + 8t^3 + 9t^4 + 12t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 3$ $0 \rightarrow A(-2)^3 \rightarrow A \oplus A(-1)^2 \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 1 + 4t + 8t^2 + 12t^3 + 16t^4 + 20t^5 + \dots & \text{if } r_A = 3 \\ 1 + 3t + 5t^2 + 7t^3 + 9t^4 + 11t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 3 + t$ $0 \rightarrow A(-2) \oplus A(-3) \rightarrow A \oplus A(-1) \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 1 + 3t + 7t^2 + 11t^3 + 15t^4 + 19t^5 + \dots & \text{if } r_A = 3 \\ 1 + 2t + 5t^2 + 6t^3 + 9t^4 + 10t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 3 + 2t$ $0 \rightarrow A(-3)^2 \rightarrow A \oplus A(-2) \rightarrow M \rightarrow 0$
	$h_M(t) = \begin{cases} 1 + 3t + 6t^2 + 10t^3 + 14t^4 + 18t^5 + \dots & \text{if } r_A = 3 \\ 1 + 2t + 4t^2 + 6t^3 + 8t^4 + 10t^5 + \dots & \text{if } r_A = 2 \end{cases}$ $s_M(t) = 3 + 2t + 1$ $0 \rightarrow A(-4) \rightarrow A \rightarrow M \rightarrow 0$

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KOEN DE NAEGHEL, DEPARTEMENT WNI, HASSELT UNIVERSITY, AGORALAAN GEBOUW D, B-3590 DIEPENBEEK (BELGIUM).

E-mail address, K. De Naeghel: koen.denaeghel@uhasselt.be